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A HIGH SCHOOL ALGEBRA

BY

J. W. A. YOUNG, Ph.D.
ASSOCIATE PROFESSOR OF THE PEDAGOGY OF MATHEMATICS
THE UNIVERSITY OF CHICAGO

AND

LAMBERT L. JACKSON, Ph.D.
FORMERLY PROFESSOR OF MATHEMATICS, STATE NORMAL
SCHOOL, BROCKPORT, NEW YORK

PART TWO

D. APPLETON AND COMPANY
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PREFACE

This volume is intended for the use of pupils who have done a year's work in High School Algebra. The first three chapters are a review and extension of the principal topics of the first year's work as presented in the authors' "A High School Algebra, Part One." The fourth chapter, that on logarithms, is, in its theory, merely an application of the subject of exponents. The remainder of the volume treats all of the topics covered in any standard second course in algebra, supplemented by problems applying algebra to geometry.

Each important process of algebra is immediately applied to the solving of equations. This plan serves not only to secure the pupil's interest, but reveals to him the utility of algebra.

Great pains have been taken to supply ample practice work, and the authors have given, under the more important topics, a greater number of exercises than will be required by any one class. In fact, there have been included as many exercises and problems as a text-book of reasonable size will admit.

Particular attention has been given to the grading of the exercises and problems, and, for convenience in checking the results, the exercises have been so constructed that the answers are not more complex than the purpose of the exercises actually requires. The authors have followed the criterion that every principle should be exemplified with the minimum of calculation.

Among the features that contribute to the teachableness of the book are the Historical Notes. These brief sketches, describing the origin of some of the more important topics of algebra, tend to stimulate the pupil's interest, and the accompanying biographical notes and portraits of famous mathematicians serve further to humanize the subject. No attempt has been made to give a connected account of the development of algebra even in outline; these notes will serve their purpose if they create a desire to read some standard work on the history of mathematics.

Other aids which teachers will appreciate are the inductive developments, the cross references, illustrative problems, methods of testing results, topical and logical arrangement, definite classification, the frequent reviews, and the summaries of the theoretic chapters.

The authors wish to express their gratitude for the assistance rendered by those who read the manuscript and the proof sheets. For valuable constructive suggestions in preparing the manuscript, they are indebted to Mr. Allen H. Knapp, of the Central High School, Springfield, Mass., and Mr. Julius J. H. Hayn, of the Masten Park High School, Buffalo, N. Y.; while for efficient aid in reading the proofs, they owe much to Mr. Matthew R. McCann, of the English High School, Worcester, Mass., and Mr. William H. Wentworth, of the Northwestern High School, Detroit, Mich. For the portraits of famous mathematicians reproduced in this volume, they are indebted to the generosity of Professor David Eugene Smith, of Teachers College, Columbia University, New York City, who placed at their disposal his unique collection.

THE AUTHORS.

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A HIGH SCHOOL ALGEBRA

PART TWO

CHAPTER I

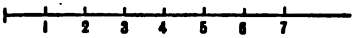
REVIEW AND EXTENSION OF PROCESSES

POSITIVE AND NEGATIVE NUMBERS

1. Algebra is concerned with the study of numbers. The number of objects in any set (for example, the number of books on a shelf) is found by counting. Such numbers are called **whole numbers** or **integers**; also **primitive** or **absolute numbers.**

In arithmetic, numbers are usually represented by means of the numerals, 0, 1, 2, 3 . . . 9, according to a system known as the **decimal notation.** In algebra, numbers are represented by numerals and also by letters, either singly or in combinations.

2. Graphical Representation. The natural integers may be represented by equidistant points of a straight line, thus:

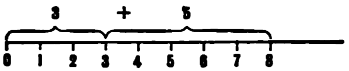


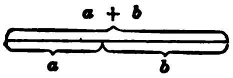
3. Addition. If two sets of objects are united into a single set (for example, the books on two shelves placed on a single shelf), the number of objects in the single set is called the sum of the numbers of objects in the two original sets. The process of finding the sum is called **addition.**

The sign, +, between two number symbols indicates that the numbers are to be added. In the simplest instances the sum is found by counting.

Thus, to find $5 + 7$, we first count 5, and then count 7 more of the number words next following (six, seven, eight, etc.). The number word with which we end (twelve) names the sum.

4. Graphical Representation. The sum of two integers may be represented graphically thus:





Theoretically, the sum of two integers can in every instance be found by counting. But it is not necessary or desirable to do so when either (or both) of the numbers is larger than nine. In this case, the properties of the decimal notation, as learned in arithmetic, enable us to abridge the process of counting.

5. Commutative Law of Addition. If two sets of objects are to be united into a single set, the number of objects in the result is obviously the same whether the objects of the second set are united with those of the first, or those of the first united with those of the second.

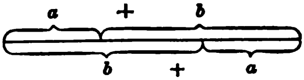
For example, the number of books is the same whether those on the first shelf be placed on the second, or those on the second be placed on the first.

In symbols: $a + b = b + a.$

This fact is called the **commutative law of addition.**

The letters a and b are here used to stand for integers, but the law will apply when they stand for any algebraic numbers.

6. Graphical Representation. The commutative law may be represented graphically thus:



7. Addition of Two or More Whole Numbers. If more than two sets of objects are united into a single set, the number of objects in the resulting set is called the **sum** of the number of objects in the original sets, and the process of finding the sum is called **addition.** As in the case of two numbers, the sum of three or more numbers may be found by counting in the simplest instances, and for larger numbers, the process may be abridged by use of the properties of the decimal notation.

8. The commutative law likewise applies to the sum of three or more integers. That is:

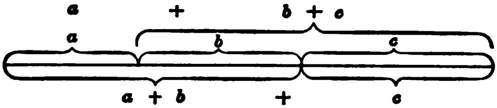
The sum is the same for every order of adding the numbers.

9. Associative Law of Addition. If we have three rows of books, the number of books is the same whether those in the second row are first placed with the first row, and then those in the third row placed with these, or those in the third row placed with the second, and then all of these with the first row.

In symbols: $(a+b)+c=a+(b+c)$.

This fact is called the **associative law of addition.**

10. Graphical Representation. The associative law may be represented graphically thus:



The properties stated above are often used to abridge calculations. Thus, $7+4+3+6$, are more easily added thus: $(7+3)+(4+6)$.

ORAL EXERCISES

Add in the easiest way:

1. $8+3+2+7$.
2. $91+43+9$.
3. $87+26+13$.
4. $13a+5a+17a+5a$.
5. $7x+12x+3x+18x$.
6. $8y+10y+7y+5y$.
7. $23a+6a+2a+4a$.
8. $48x+73x+2x+7x$.
9. $19y+54y+6y+y$.
10. $73b+186b+14b$.
11. $279t+347t+21t$.
12. $624p+45p+6p+5p$.
13. $93t+9t+7t+t$.
14. $144m+7m+6m+3m$.

WRITTEN EXERCISES

Show graphically that:

1. $11+4+6=11+(4+6)$.
2. $3+(4+1)=1+3+4$.
3. $8+5=5+8$.
4. $4a+5b=5b+4a$.

11. Subtraction. It often happens that we wish to know how many objects are left when some of a set are taken away, or to know how much greater one number is than another. The process of finding this number is called **subtraction.** The number taken away is called the **subtrahend,** that from which it is taken, the **minuend,** and the result, the **difference** or the **remainder.**

12. The sign of subtraction is —.

13. Subtraction is the reverse of addition, and from every sum one or more differences can at once be read.

Thus, from $5 + 7 = 12$ we read at once $12 - 5 = 7$,
$12 - 7 = 5$.
And from $5 + 5 = 10$, we read $10 - 5 = 5$.
And from $a + b = c$, we read $c - a = b$,
$c - b = a$.
Likewise, from $a + b + c = d$ we read $d - a = b + c$,
$d - (a + b) = c$, etc.

14. There is no commutative law of subtraction. For $7 - 4$ is not the same as $4 - 7$. In fact, the latter indicated difference has no meaning in arithmetic. We cannot take a larger number of objects from a smaller number.

15. In algebra, where numbers are often represented by letters, we may not know whether the minuend is larger than the subtrahend or not. For example, in $a - b$, we do not know whether a is larger than b or not. But it is desirable that such expressions should have a meaning in all cases, and this is accomplished by the definition and use of **relative numbers.**

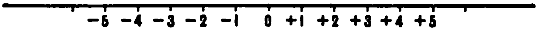
16. The First Extension of the Number System. Relative Numbers. Whenever quantities may be measured in one of two opposite senses such that a unit in one sense offsets a unit in the other sense, it is customary to call one of the senses the positive sense, and the other the negative sense, and numbers measuring changes in these senses are called **positive** and **negative** numbers respectively. (Examples, see Chap. IV, Pt. I.)

17. A number to be added is offset by an equal number to be subtracted; hence such numbers satisfy the above definition, and numbers to be added are called positive, and those to be subtracted are called negative. Consequently, positive and negative numbers are denoted by the signs + and − respectively.

Thus, + 5 means positive five, and denotes five units to be added or to be taken in the positive sense.

− 5 means negative five, and denotes five units to be subtracted, or to be taken in the negative sense.

18. Graphical Representation. Relative integers may be represented graphically thus:



It appears that the positive integers are represented by just the same set of points as the natural or absolute integers. For this and other reasons the absolute numbers are usually identified with positive numbers. Although it is usually convenient to do this, we have in fact the three classes of numbers: the absolute, the positive, and the negative. Thus, we may consider $5 without reference to its relation to an account, or we can consider it as $5 of assets, or we may consider it as $5 of debts.

19. According to Sec. 17, the signs +, − denote the operations of addition or subtraction, or the positive or negative character of the numbers which these signs precede.

If it is necessary to distinguish a sign of character from a sign of operation, the former is put into a parenthesis with the number it affects.

Thus, $+8-(-3)$, means: positive 8 minus negative 3.

When no sign of character is expressed, the sign plus is understood.

Thus, $5-3$ means: positive 5 minus positive 3.
Similarly, $8a+9a$ means: positive $8a$ plus positive $9a$.

20. Absolute Value. The value of a relative number apart from its sign is called its **absolute value.**

ORAL EXERCISES

Read the following in full, according to Sec. 19:

1. $6-4$.
2. $-5-8$.
3. $-8+20$.
4. $2a+3a$.
5. $2b-3b$.
6. $-2a-3a$.
7. $2y+(+3y)$.
8. $3p-(-2p)$.
9. $12-(-5)$.
10. $-12-(+5)$.
11. $-7-(-9)$.
12. $2a-(+3a)$.
13. $c+d$.
14. $c-d$.
15. $m+(-n)$.
16. $4x+(-2x)$.
17. $7-9$.
18. $7+9$.
19. $-7+(-9)$.
20. $2y-(-3y)$.
21. $-2x-(-3x)$.
22. $-2x+(-3x)$.
23. $-2b-(-5c)$.
24. $3a-(+5y)$.

WRITTEN EXERCISES

Indicate, using the signs $+$, $-$:

1. The sum of positive 5 and positive 3.
2. The sum of positive a and negative b.
3. The difference of positive p and positive q.
4. The difference of negative 5 and positive 3.
5. The difference of negative x and positive y.
6. The sum of positive a and positive b.
7. The sum of negative ab and negative ab.
8. The sum of positive y and negative x.
9. The difference of positive xy and negative xy.
10. The difference of negative pq and positive mn.

21. Addition of Relative Numbers.

Just as 3 pounds + 5 pounds = 8 pounds,
so 3 positive units + 5 positive units = 8 positive units,
and 3 negative units + 5 negative units = 8 negative units.

To add units of opposite character, use is made of the defining property of relative numbers, that a unit in one sense offsets a unit in the other sense. Thus, to add 3 positive units and 7 negative units we notice that the 3 positive units offset

3 of the negative units and the result of adding the two will be 4 negative units.

That is, $(+3)+(-7)=(+3)+(-3)+(-4)=-4$.

In general:

I. *If two relative numbers have the same sign, the absolute value of the sum is the sum of the absolute values of the addends, and the sign of the sum is the common sign of the addends.*

II. *If two relative numbers have opposite signs, the absolute value of the sum is the difference of the absolute values of the addends, and the sign of the sum is the sign of the addend having the larger absolute value.*

22. More than two numbers are added by repetition of the process just described. This may be done either:

(1) *by adding the second number to the first; then the third number to the result, and so on;* or

(2) *by adding separately all the positive numbers and all the negative numbers, and then adding these two results.*

23. It may be verified that the Commutative and the Associative Laws of Addition hold also for relative integers.

24. Subtraction of Relative Numbers. Since n units of one sense are offset by adding n units of the opposite sense, we may subtract n units of one sense by adding n units of the opposite sense.

Thus, $7-(+3)=7+(-3)$.

And, $7-(-3)=7+(+3)$.

And, $4-(+7)=4+(-7)$.

25. Accordingly, subtraction may be regarded as the inverse of addition: *To subtract a monomial, we add its opposite.*

To subtract an algebraic expression consisting of more than one term, we subtract the terms one after another.

In general, *to subtract any algebraic expression we may change the sign of each of its terms and add the result to the minuend.*

ORAL EXERCISES

State the sums:

1. $5+(-3)$.
2. $-6a+(-7a)$.
3. $-11y+3y$.
4. $-12z+(-18z)$.
5. $11x+(-2x)+(-5x)$.
6. $-3q+7q+(-6q)$.

7. How may the correctness of a result in subtraction be tested? State the differences:

8. $11-6$.
9. $-11-6$.
10. $-11a-(-6a)$.
11. $31x-(+5x)$.
12. $-31y-(-3y)$.
13. $17p-(-17p)$.

14. How may a parenthesis preceded by the sign + be removed without changing the value of the expression? One preceded by the sign − ?

15. How may terms be introduced in a parenthesis preceded by the sign + without changing the value of the expression? In a parenthesis preceded by the sign − ?

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} 2a+5 \\ \underline{a+4} \end{array}$
2. $\begin{array}{r} 3a+8 \\ \underline{a-4} \end{array}$
3. $\begin{array}{r} 6b+c \\ \underline{3b-2c} \end{array}$
4. $\begin{array}{r} -3a+b \\ \underline{2a-3b} \end{array}$
5. $\begin{array}{r} 4a-5 \\ \underline{3a+7} \end{array}$
6. $\begin{array}{r} c+d-5 \\ \underline{c-d+5} \end{array}$
7. $\begin{array}{r} x+y+2z \\ \underline{x-y+4z} \end{array}$
8. $\begin{array}{r} p+q-m \\ \underline{p-q+2m} \end{array}$
9. $\begin{array}{r} 2x-y+z \\ \underline{2x+2y-4z} \end{array}$
10. $\begin{array}{r} ax+by+c \\ \underline{ax+y-c} \end{array}$
11. $\begin{array}{r} 4x-2z+y \\ \underline{2x-y+z} \end{array}$
12. $\begin{array}{r} 1+m^3+p^2 \\ \underline{1-m^3-p^2} \end{array}$
13. $\begin{array}{r} ax+by+cz^2 \\ \underline{bx+ay-z^2} \end{array}$
14. $\begin{array}{r} 1.5x+3.5y+z \\ \underline{.5x+6.5y-.1z} \end{array}$
15. $\begin{array}{r} \frac{1}{2}x+\frac{4}{5}y-\frac{1}{3}z \\ \underline{\frac{1}{4}x-\frac{3}{5}y+\frac{2}{3}z} \end{array}$

Subtract:

16. $\begin{array}{r} 4a+6 \\ \underline{2a-9} \end{array}$
17. $\begin{array}{r} 8x+3 \\ \underline{-3x+2} \end{array}$
18. $\begin{array}{l} 11x-4y \\ \underline{19x} \end{array}$

19. $\begin{array}{r} 12t \\ -6t+3 \\ \hline \end{array}$

20. $\begin{array}{r} 7x+3y \\ 2y-4x \\ \hline \end{array}$

21. $\begin{array}{r} 4a+3b \\ 2c-5a \\ \hline \end{array}$

22. $\begin{array}{r} 16y+-z \\ 8y-10z \\ \hline \end{array}$

23. $\begin{array}{r} 10x^2-16y^2 \\ 20x^2+\ 4y^2 \\ \hline \end{array}$

24. $\begin{array}{r} 41x^2-16y^2 \\ 15x^2-20y^2 \\ \hline \end{array}$

25. $\begin{array}{r} -11m+40p \\ -40m-12p \\ \hline \end{array}$

26. $\begin{array}{r} x-7y+5z \\ -x+4y-6z \\ \hline \end{array}$

27. $\begin{array}{r} p+\ q-\ m \\ 6p-2q+4m \\ \hline \end{array}$

28. $\begin{array}{l} ax^2+by^2+c \\ ax^2-by^2-c \\ \hline \end{array}$

29. $\begin{array}{r} m^2+2p^2-6q^2 \\ -m^2-4p^2+5q^2 \\ \hline \end{array}$

30. $\begin{array}{l} 40x^2-10y^2+\ z^2 \\ 50x^2+40y^2-7z^2 \\ \hline \end{array}$

31. $\begin{array}{l} 2.5a+6.3b-.1c \\ 1.5a-3.5b-.9c \\ \hline \end{array}$

32. $\begin{array}{l} \frac{1}{2}x-\frac{2}{3}y+\frac{1}{5}z \\ \frac{3}{4}x-\frac{1}{3}y+\frac{4}{5}z \\ \hline \end{array}$

33. $\begin{array}{l} a^2x+b^2y+c^2z^2 \\ a^2x+by^2+cz^2 \\ \hline \end{array}$

Remove parentheses and unite terms as much as possible:

34. $3a-2b+(3b-7a)$.

35. $4m+5-(63-3p)$.

36. $(11x+5y)-(-3x+2z)$.

37. $7-[2-(3-5)]$.

38. $(4a+2a)-[(7a-5a)+(-6a-17a)]$.

39. $3+\{5x-2-(7x-1-2+3x)\}$.

40. $4x+\{2x-[3-(7x+5)-1+x]\}$.

41. $ab-\{2ab+c-[3c-(6-ab)]\}$.

42. $12xy-\{2xy-6z+[4xy-(2z+xy)]\}$.

43. $-(5pq-3xy+1)-\{-2pq+3xy-(xy+2)\}$.

44. $-\{4abc-(2ac+bc)\}+\{6abc+(2ac-bc)\}$.

Write the expressions of Exercises 45–56, as x minus a parenthesis. Also group the terms involving x and y in each expression in a parenthesis preceded by the sign $-$.

45. $3a+x+2y$.

46. $-5y+7c-8+x$.

47. $6t^2+4y+x-3$.

48. $2m+p+x-y$.

49. $a+x-b+cy$.

50. $p+x-y+by$.

51. $x - 3b + 2y - c.$

52. $3b + x + ay + by.$

53. $by + m - n + x.$

54. $2p - q + x - y.$

55. $ay + x - z + zy.$

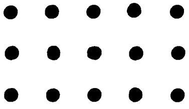
56. $c + d + x - (a + b)y.$

26. Multiplication. To multiply two absolute integers means to use the one (called the **multiplicand**) as addend as many times as there are units in the other (called the **multiplier**). The result is called the **product**.

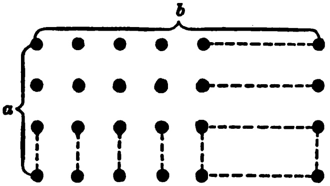
Thus, 3 times 4 means $4 + 4 + 4$.

The simpler products are obtained by actually making the additions that are implied. For large numbers, as we have seen in arithmetic, the process may be much abridged by use of properties of the decimal notation.

27. Commutative Law of Multiplication. The expression 3 times 5 means $5 + 5 + 5$, and may be indicated as follows:



That is, since each horizonal line (or *row*) contains 5 dots, there are all together 3 times 5 dots. But each vertical line (or *column*) contains 3 dots and there are 5 columns. Hence there are 5 times 3 dots all together. But the number of dots is the same whether we count them by rows or by columns, hence *5 times 3 equals 3 times 5.* Quite similarly, if we have a rows of dots with b dots in each row, it follows that a times b equals b times a.



The result may be stated in symbols thus:

$$ab = ba.$$

This fact, called the **Commutative Law of Multiplication**, means that the product is not altered if multiplier and multiplicand are interchanged. Consequently, these names are frequently replaced by the name **factor** applied to each of the numbers multiplied.

28. Let each dot of the block of dots of (*A*) above have the value of 6. Since there are 3 × 5 dots, the value of the block would be (3 × 5) × 6.

A second expression for the value of the block is obtained by finding the value of the first row (viz. 5 × 6) and multiplying it by the number of rows, or 3. The expression resulting must be equal to that already found, or:

$$3 \times (5 \times 6) = (3 \times 5) \times 6.$$

Similarly, if each dot of the block (*B*) above has the value c, it follows that

$$a(bc) = (ab)c$$

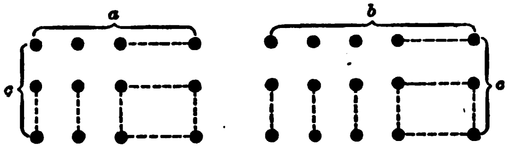
That is: *The product of three absolute integers is not altered if they be grouped for multiplication in any way possible without changing the order. This is a case of what is known as the* **Associative Law of Multiplication** *for absolute integers.*

29. By similar methods and use of these results it can be proved that both the Commutative Law and the Associative Law apply to all products of absolute integers. That is:

Commutative Law. *The product of any number of given factors is not changed, if the order of the factors be changed in any way.*

Associative Law. *The product of any number of given factors in a given order is not changed if the factors be grouped in any way.*

The Distributive Law. From the block of dots we see that



$$c(a+b) = ca + cb.$$

This is called the **Distributive Law** of Multiplication, and the above proof covers the case in which a, b, and c are absolute integers.

30. Multiplication of Relative Integers. To multiply by a positive integer means to take the multiplicand as addend as many times as there are units in the multiplier, and to multiply by a negative integer means to take the multiplicand as subtrahend as many times as there are units in the multiplier.

Consequently,

$$4 \cdot 3 = 3 + 3 + 3 + 3 = 12.$$
$$4(-3) = -3 + (-3) + (-3) + (-3) = -12.$$
$$(-4)3 = -3 - 3 - 3 - 3 = -12.$$
$$(-4)(-3) = -(-3) - (-3) - (-3) - (-3) = 12.$$

So, generally,

$$(+a)(+b) = +ab,$$
$$(+a)(-b) = -ab,$$
$$(-a)(+b) = -ab,$$
$$(-a)(-b) = +ab.$$

31. In words: *The product of two (integral) factors of like signs is positive, and of two factors of unlike signs is negative; in each case the absolute value of the product is the product of the absolute values of the factor.*

32. We observe that the Commutative Law holds in this case also.

For example: $(-b)a = a(-b)$. Since $ba = ab$, by the Commutative Law for absolute integers, and by Sec. 30, $(-b)a = -ba = -ab = a(-b)$.

It may be shown that the Associative and the Distributive Laws also hold.

ORAL EXERCISES

State the laws that are applied in the various steps of the following calculations:

1. $2 \cdot 8 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 8 \cdot 3 = 10 \cdot 24 = 240.$

2. $5(17 \cdot 2 - 6c) = 5(17 \cdot 2) - 5(6c) = (5 \cdot 2)17 - (5 \cdot 6)c$
$= 10 \cdot 17 - 30c = 170 - 30c.$

3. $7b(5x + ab) = (7b)(5x) + (7b)(ab) = 35bx + 7ab^2.$

4. $(a+b)(c+d)=a(c+d)+b(c+d)=ac+ad+bc+bd.$

State the products:

5. $\begin{array}{r} ax \\ \underline{5b} \end{array}$

6. $\begin{array}{r} -2a \\ \underline{-3a^2} \end{array}$

7. $\begin{array}{r} 8x \\ \underline{-4xy} \end{array}$

8. $\begin{array}{r} a^5b \\ \underline{a^2c} \end{array}$

9. $\begin{array}{r} 5x^2 \\ \underline{2x^3} \end{array}$

10. $\begin{array}{r} -x^2y \\ \underline{-xy^2} \end{array}$

11. $\begin{array}{r} 6xyz \\ \underline{-2xz^3} \end{array}$

12. $\begin{array}{r} -4a^2t \\ \underline{-3b^2t} \end{array}$

13. $\begin{array}{r} -5x^3y^2 \\ \underline{5x^2y^3} \end{array}$

14. $(a+b)^2.$

15. $(a-c)^2.$

16. $(x+y)^2.$

17. $(x-a)^2.$

18. $(x-2y)^2.$

19. $(x+y)(x-y).$

20. $(2x-1)^2.$

21. $(2m+n)^2.$

22. $(x-1)(x+1).$

23. $(1-y)(1+y).$

24. $(m+2x)^2.$

25. $(3-2d)(3+2d).$

26. $(4-x)^2.$

27. $-3x(x+2).$

28. $(1-2d)(1+2d).$

29. $(\frac{1}{2}x-\frac{1}{3}y)(\frac{1}{2}x+\frac{1}{3}y).$

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{r} 2a+3 \\ \underline{2a+4} \end{array}$

2. $\begin{array}{r} 3a-2b \\ \underline{2a-3b} \end{array}$

3. $\begin{array}{r} x+2 \\ \underline{x-3} \end{array}$

4. $\begin{array}{r} x^2-3x+1 \\ \underline{x-2} \end{array}$

5. $\begin{array}{r} y^2-3y+4 \\ \underline{y-2} \end{array}$

6. $\begin{array}{r} p-3t+t^3 \\ \underline{p-2t} \end{array}$

7. $(3x-4y)^2.$

8. $(5y-7t)(8y+2t).$

9. $(6a+13q)^2.$

10. $(1+x)(2+x)(1-x).$

11. $(1-2y)^2.$

12. $(b-3)(b+7)(b-3).$

13. $(x^n+y^n)(x^n-y^n).$

14. $(a^p-b^m-1)^2.$

15. $(4x-3xy)(2x+2y^2).$

16. $(ax+by)(cx+dy).$

17. $(ax^2-by)(cx-dy^2).$

18. $(m+n+p)(m+n-p).$

19. $(a+b+c-d)^2.$

20. $(a-2b)(a-2b)(a-2b).$

33. Detached Coefficients. — When two polynomials can be arranged in descending powers of the same letter, the work of multiplication may often be shortened by *using the coefficients only.* This is called **multiplication by detached coefficients.**

Thus, to multiply $3x^2-2x+1$ by $x-1$:

WORK IN FULL	WORK WITH COEFFICIENTS ONLY	TEST
$3x^2-2x+1$	$3-2+1$	$x=1 \mid 2$
$\underline{x-1}$	$\underline{1-1}$	$\underline{0}$
$3x^3-2x^2+x$	$3-2+1$	
$\underline{-3x^2+2x-1}$	$\underline{-3+2-1}$	
$3x^3-5x^2+3x-1$	$3-5+3-1$	0

The processes in the two cases are identical with the exception of the omission of the letters in the second case. The product of $3x^2$ and x, the first terms of the factors, show that the first term of the product is $3x^3$, the next term must contain x^2 and the next x, because the terms in the result must be in order of degree. Thus we may perform the operation with coefficients, and then supply the proper letters and exponents.

When some powers of the letters are missing, zeros must be supplied as coefficients of the missing powers in order to keep a record of the places in which the powers are missing.

Thus, to multiply $2x^3-7x+3$ by $2x-5$, write $2x^3-7x+3$ as $2x^3+0x^2-7x+3$, and perform the multiplication as shown at the right.

$$\begin{array}{r|r}
 & \text{TEST} \\
2+0-7+3 & -2 \\
\underline{2-5} & \underline{-3} \\
4+0-14+6 & \\
\underline{-10-0+35-15} & \\
4-10-14+41-15 & +6
\end{array}$$

$\therefore (2x-5)(2x^3-7x+3)=4x^4-10x^3-14x^2+41x-15.$

WRITTEN EXERCISES

Multiply by detached coefficients and test:

1. $\begin{array}{l} m+n \\ \underline{2m-3n} \end{array}$

2. $\begin{array}{l} x^3-5x+3 \\ \underline{x-1} \end{array}$

3. $\begin{array}{l} a^2+2a+1 \\ \underline{a+1} \end{array}$

4. $\begin{array}{l} x^4-3x^3+5 \\ \underline{x+2} \end{array}$

5. $\begin{array}{l} y^3-y^2+5 \\ \underline{2y+3} \end{array}$

6. $\begin{array}{l} x^4+2x^2+7x \\ \underline{2x-1} \end{array}$

7. x^2-x-6 by $2x^2-4x-16$.

8. a^2-a-12 by $3a^2-15a+18$.

9. $a^4 - 2a^2 + 1$ by $a^4 + 2a^2 + 1$.
10. $m^3 - m^2 - m + 1$ by $2m - 1$.
11. $1 - x - x^2$ by $2 + x^3 - x^2 + 2x$.
12. $x^2 + xy + y^2$ by $x^2 + y^2 - xy$.
13. $x^{2a} - 4x^a + 4$ by $1 - x^a + x^{3a} - 2x^{2a}$.

34. Division. Division is the process of finding a number called the **quotient**, such that when multiplied by a given number, called the **divisor**, the product is a given number, called the **dividend.**

35. The fundamental relation of division is:

Divisor times Quotient equals Dividend.

From this it follows at once that if dividend and divisor have the same sign, the quotient is positive, and if they have unlike signs, the quotient is negative.

36. In dividing polynomials, it is best to arrange both dividend and divisor in the order of the powers of the same letter.

Thus, $(x^3 - 3x^2 + 3x - 1) \div (x - 1)$.
instead of $(3x - 3x^2 + x^3 - 1) \div (x - 1)$.

WRITTEN EXERCISES

Divide:

1. $x^3 + x^2 - x - 1$ by $x + 1$.
2. $x^4 - 1$ by $x - 1$.
3. $y^3 - z^3$ by $y^2 + yz + z^2$.
4. $x^{3a} - y^{3a}$ by $x^a - y^a$.
5. $x^6 - 1$ by $x^2 - 1$.
6. $p^5 + q^5$ by $p + q$.
7. $m^3 + m^2 + m + 1$ by $m + 1$.
8. $x^2 - px - qx + pq$ by $x - p$.
9. $4x^4 + 1$ by $2x^2 - 2x + 1$.
10. $x^2 - 2xy - 3y^2$ by $x - 3y$.
11. $6a^3 - 10a^2 + 13a - 6$ by $3a - 2$.
12. $6m^4 - 2m^3 - 8m^2 + 4m - 8$ by $3m^2 - m + 2$.
13. $x^{2a} + x^a - 6$ by $x^a - 2$.
14. $x^{4a} + 4x^{3a}y + 6x^{2a}y^2 + 4x^ay^3 + y^4$ by $x^{2a} + 2x^ay + y^2$.

37. Detached Coefficients. When both divisor and dividend involve a regular series of powers of the same letters, it is easier to divide with *coefficients only.*

Thus, to divide $x^3 - 4x^2 + 4x - 1$ by $x - 1$.

WORK IN FULL

$$\begin{array}{r|l} & x^2 - 3x + 1 \\ \hline x - 1 & x^3 - 4x^2 + 4x - 1 \\ & \underline{x^3 - x^2} \\ & -3x^2 \\ & \underline{-3x^2 + 3x} \\ & x - 1 \\ & \underline{x - 1} \end{array}$$

WORK WITH COEFFICIENTS ONLY

$$\begin{array}{r|l} & 1 - 3 + 1 \text{ or } x^2 - 3x + 1 \\ \hline 1 - 1 & 1 - 4 + 4 - 1 \\ & \underline{1 - 1} \\ & -3 \\ & \underline{-3 + 3} \\ & 1 - 1 \\ & \underline{1 - 1} \end{array}$$

TEST. Letting $x = 2$,

$$(2 - 1)(4 - 6 + 1) = 8 - 16 + 8 - 1.$$
$$-1 = -1.$$

38. If the series of powers is not complete, zero coefficients must be used.

Thus, $x^3 + 3x^2 + 1$ must be regarded as $x^3 + 3x^2 + 0x + 1$, and the coefficients $1 + 3 + 0 + 1$ must be used in the division.

WRITTEN EXERCISES

Divide by detached coefficients and test:

1. $\dfrac{a^4 - 1}{a - 1}$.
2. $\dfrac{a^3 - 1}{a^2 + a + 1}$.
3. $\dfrac{8x^3 + 1}{2x + 1}$.
4. $(x^2 + 5x + 6) \div (x + 2)$.
5. $(x^2 - 2x - 3) \div (x - 3)$.
6. $(10 + x^2 - 7x) \div (x - 2)$.
7. $(x^4 - 1) \div (x^2 - 1)$.
8. $(x^3 + 3x^2 + 3x + 1) \div (x + 1)$.
9. $(y^3 - 3y^2 + 3y - 1) \div (y - 1)$.
10. $(6x + 1 + 9x^2) \div (3x + 1)$.
11. $(a^4 + a^2 + 1) \div (a^2 - a + 1)$.
12. $\dfrac{m^4 + 4m^3 + 6m^2 + 4m + 1}{m^2 + 2m + 1}$.

39. Special Quotients. The general form of quotients represented by the type $(x^n \pm y^n) \div (x \pm y)$ may readily be seen by division.

Thus, considering $(x^n + y^n) \div (x + y)$, we have

$$\begin{array}{r|l} & x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots \\ \hline x + y & x^n \qquad\qquad + y^n \\ & \underline{x^n + x^{n-1}y} \\ & \quad - x^{n-1}y \\ & \quad \underline{- x^{n-1}y - x^{n-2}y^2} \\ & \qquad\qquad + x^{n-2}y^2 + \cdots \end{array}$$

By inspection, what do you think the fourth term of the quotient will be? The fifth? Verify your opinion by continuing the above division.

WRITTEN EXERCISES

Find the quotient and remainder if any:

1. $(x^5 - 1) \div (x - 1)$.
2. $(x^6 + 1) \div (x - 1)$.
3. $(x^5 + 1) \div (x + 1)$.
4. $(x^6 - 1) \div (x + 1)$.
5. $(x^4 - y^4) \div (x + y)$.
6. $(x^4 - y^4) \div (x - y)$.
7. $(x^5 + y^5) \div (x + y)$.
8. $(x^7 + y^7) \div (x + y)$.
9. $\dfrac{x^7 - y^7}{x - y}$.
10. $\dfrac{x^7 + y^7}{x - y}$.
11. $\dfrac{x^6 - y^6}{x - y}$.
12. $\dfrac{x^6 + y^6}{x - y}$.
13. $\dfrac{x^6 - y^6}{x + y}$.
14. $\dfrac{x^6 + y^6}{x + y}$.

15. Substitute 5 for n in the first five terms of the above division, and compare with the result of Exercise 7.

16. Substitute 6 for n similarly in the first six terms, and compare with the result of Exercise 14.

17. Use $n = 7$ in the first 7 terms; compare with Exercise 8.

18. Take $n = 5$ and $y = 1$, and compare with Exercise 3.

19. Divide $x^n + y^n$ by $x - y$ to seven terms. Let $n = 6$ and compare with the result in Exercise 12. Let $n = 7$ and compare with the result in Exercise 10.

20. Divide $x^n - y^n$ by $x + y$ to six terms. Let $n = 6$ and compare with the result of Exercise 13.

21. Divide $x^n - y^n$ by $x - y$ to six terms. Let $n = 6$ and compare with the result of Exercise 11.

ZERO AND ITS RELATION TO THE PROCESSES

40. Definition of Zero. Zero may be defined as the result of subtracting a number from itself.

$$a - a = 0.$$

41. Addition. By definition of zero, $a + 0 = a + b - b = a$, since to add b and immediately take it away leaves the original number a.

$$a + 0 = a.$$

42. Subtraction. Similarly, $a - 0 = a - (b - b) = a - b + b = a$, since to take away b and then replace it leaves the original number a.

$$a - 0 = a.$$

To add or subtract zero does not alter the original number.

43. Multiplication. By definition of zero,

$$0 \cdot a = (b - b)a = ba - ba = 0.$$

Or, $$0 \cdot a = 0.$$

That is, if one factor is zero, the product is zero.

Multiplication by zero simply causes the multiplicand to vanish.

It follows directly that $\frac{0}{a} = 0$.

44. Division. According to the definition of division $a \div 0$, or $\frac{a}{0}$, asks: By what must zero be multiplied to produce a?

Let x denote the desired number. Then $0 \cdot x = a$.

But we know that zero times any number is zero. If a is not zero, there is no number x that satisfies the above equation.

That is, $\frac{a}{0} =$ **no number.**

If a is zero, every number x satisfies the equation.

That is, $\frac{0}{0} =$ **any number,** since 0 times any number $= 0$.

Division by zero is therefore either entirely indefinite or impossible. In either case it is not admissible.

If we divide one literal expression by another, there is no guarantee that the result is correct for those values of the letters that make the divisor zero.

EXAMPLE

Let $a = b.$ (*1*)
Multiplying both members by a, $a^2 = ab.$ (*2*)
Subtracting b^2 from both members, $a^2 - b^2 = ab - b^2$ (*3*)
Factoring, $(a + b)(a - b) = b(a - b).$ (*4*)
Dividing both members by $a-b$, $a + b = b.$ (*5*)
Substituting the value of a from (*1*), $b + b = b.$ (*6*)
Or, $2b = b.$ (*7*)
Dividing by b, $2 = 1.$ (*8*)

The work is quite correct to equation (*4*). But by dividing equation (*4*) by an expression that, according to the conditions of the problem, is zero, we find as the result an incorrect equation.

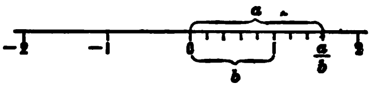
FRACTIONS

45. The Second Extension of the Number System; Fractions. We found the primitive or absolute integers by counting. We defined the operation of addition for these integers and saw that it was always possible. Next we defined the operation of subtraction for these integers, and found that it was not always possible. This led us to define relative numbers (positive and negative). In the system of numbers as enlarged by this first extension, we saw that both addition and subtraction are always possible. Then we defined multiplication for all integers and saw that it was always possible. We now examine the operation of division as just defined.

The operation $12 \div 4$ is possible in the system of integers because there exists an integer, 3, whose product with 4 is 12. But the operation $12 \div 5$ is impossible in the system of integers, since there exists no integer whose product with 5 is 12. This leads us to define another kind of number, the fraction. This is done by dividing the unit into b equal parts, and taking a of these parts. A symbol for the new number is $\frac{a}{b}$, in which a and b may be any integers. This constitutes our **second extension** of the number system.

46. The new number, $\frac{a}{b}$, is called a **fraction**; a is called the **numerator** and b the **denominator** of the fraction; a and b together are called the **terms** of the fraction.

47. Graphical Representation of Fractions. Fractions may be represented by points of the number scale that we have already had.



Thus, if the distance from *zero* to 1 is divided into b equal parts, and then a of these parts laid off from *zero*, the end point represents the fraction $\frac{a}{b}$. (In the figure the fraction is $\frac{8}{5}$.) Fractions may also be taken in the negative sense. Two fractions are said to be equal, or to have the same value, when they are represented by the same point of the number scale.

48. If each of the b equal parts is halved, making $2b$ parts, and each of the a parts taken is halved, making $2a$ parts, the end point remains the same.

That is:

$$\frac{a}{b}=\frac{2a}{2b}.$$

Similarly:

$$\frac{a}{b}=\frac{na}{nb}.$$

(*A*) In words: *The value of a fraction is not altered if both numerator and denominator is multiplied by the same integer.*

(*B*) Reading the above equation from right to left: *The value of a fraction is not altered if both numerator and denominator be divided by a common integral factor.*

Every integer may be regarded as a fraction.

Thus, $3=\frac{3}{1}$ or $\frac{6}{2}$ or $\frac{15}{5}$ etc. $a+b=\frac{a+b}{1}$.

49. Addition and Subtraction of Fractions. The properties (*A*) and (*B*), Sec. 48, enable us to add and subtract fractions.

If fractions have not the same denominator, they are first reduced to the same denominator, and then the results added.

WRITTEN EXERCISES

Add and express the results in lowest terms:

1. $\frac{3}{5}+\frac{1}{10a}$.
2. $\frac{4a}{5}-\frac{5a}{4}$.
3. $\frac{3}{b-1}-\frac{2}{b+1}$.
4. $\frac{x-1}{x+1}-\frac{x+4}{2(x+1)}$.
5. $\frac{x^2}{2y}+\frac{3y}{4x}$.
6. $\frac{5}{a-b}+\frac{2}{b}$.
7. $\frac{2}{a+p}-\frac{3p}{(a+p)^2}$.
8. $\frac{5}{x^4}+\frac{3}{2x^2}-1$.
9. $\frac{2q}{1-q^2}+\frac{5}{1+q}$.
10. $\frac{x}{(1-x)(1-y)}+\frac{1}{(y-x)(y-1)}+\frac{y}{(x-1)(1-y)}$.
11. $\frac{1}{a^2-(b+c)a+bc}+\frac{1}{a^2-(b+d)a+bd}+\frac{1}{a^2-(c+d)a+cd}$.
12. $\frac{2xy+y^2}{(x-y)^2}-\frac{y}{x-y}-\frac{x^2+5xy}{(x+y)^2}$.

50. The addition and reduction of fractions is facilitated by the processes for finding the l.c.m. and the h.c.f. The methods for finding these by factoring given in Chapter XIV, Part I, are sufficient for elementary algebra.

WRITTEN EXERCISES

Find the l.c.m. of:

1. 48, 60, 120.
2. $15a^2$, $21a^2x$, $105abxy$.
3. $5abx$, $25a^2x$, $40b^2x^2$.
4. x^2-x, x^2-2x+1.
5. x^2-5x-6, x^2+6x+5.
6. $a(x-1)$, $b(1-x^2)$, $c(x-x^2)$.
7. $(1-3x)^2$, $1-9x^2$, $1-4x+3x^2$.
8. m^2+m+1, m^2-m+1, m^4+m^2+1.
9. $5x^2-18x+9$, $4x^2-11x-3$.
10. $x-1$, $(x-1)^3$, x^2-2x+1.
11. $(p-q)(p+r)$, $(q-r)(q-p)$, $(r-q)(r+p)$.
12. $(a-c)(a-b)$, $(c-a)(b-c)$, $(b-a)(c-b)$.
13. $x^3-6x^2+11x-6$, x^2-4x+3, x^2-3x+2.

Find the h. c. f. of:

14. 28, 35, 105. **16.** $1-a,\ a^2-2a+1$.

15. $21x^2,\ 42ax^2,\ 63abx^3y$. **17.** $1-x^3,\ 1-x^2,\ x^2-2x+1$.

18. $a^2+2ab+b^2,\ a^5+b^5$.

19. $x+y,\ x^3-xy^2,\ x^2-2xy-3y^2$.

20. $x^4+x^2y^2+y^4,\ x^2+y^2-xy$.

21. $a^3+5a^2-6a,\ 2a^2-2$.

22. $3z^3-7z^2+4,\ 5z^3-17z^2+16z-4$.

23. $a^2+b^2+2c^2+2ab+3ac+3bc,\ ac+bc+c^2$.

51. Fractions may sometimes be added more easily by adding them in an order different from that in which they were given.

EXAMPLES

1. In adding $\dfrac{1}{a-1}+\dfrac{1}{a-3}+\dfrac{-1}{a+1}+\dfrac{1}{a+3}$ it is especially easy to add the first and third fractions; then the second and fourth; and, finally, the results thus obtained.

Thus,

$$\frac{1}{a-1}-\frac{1}{a+1}=\frac{a+1-a+1}{a^2-1}=\frac{2}{a^2-1}.$$

$$\frac{1}{a-3}+\frac{1}{a+3}=\frac{a+3+a-3}{a^2-9}=\frac{2a}{a^2-9}.$$

$$\frac{2}{a^2-1}+\frac{2a}{a^2-9}=\frac{2a^2-18+2a^3-2a}{(a^2-1)(a^2-9)}=\frac{2(a^3+a^2-a-9)}{(a^2-1)(a^2-9)}.$$

2. In adding $\dfrac{3}{a-b}+\dfrac{3}{a+b}+\dfrac{6a}{a^2+b^2}+\dfrac{12a^3}{a^4+b^4}$ the first and second are easily combined, then that result is easily added to the third, and finally that result to the fourth.

(1) $$\frac{3}{a-b}+\frac{3}{a+b}=\frac{6a}{a^2-b^2}.$$

(2) $$\frac{6a}{a^2-b^2}+\frac{6a}{a^2+b^2}=\frac{12a^3}{a^4-b^4}.$$

(3) $$\frac{12a^3}{a^4-b^4}+\frac{12a^3}{a^4+b^4}=\frac{24a^7}{a^8-b^8}.$$

WRITTEN EXERCISES

1. Add the fractions in the examples above in the ordinary way; that is, find the l. c. d. of the several fractions and add at once.

Add by successive combinations, as above:

2. $\dfrac{1}{a-x}+\dfrac{1}{a+x}-\dfrac{2a}{a^2+x^2}.$

3. $\dfrac{1}{a-1}+\dfrac{3}{a-2}-\dfrac{3}{a+2}-\dfrac{1}{a+1}.$

4. $\dfrac{2}{a-b}+\dfrac{2}{a+b}+\dfrac{4a}{a^2+b^2}.$

5. $\dfrac{1}{a-b}+\dfrac{a-b}{c}-\dfrac{1}{a+b}-\dfrac{a+b}{c}.$

6. $\dfrac{1}{p+1}-\dfrac{2}{p-2}+\dfrac{2}{p+2}-\dfrac{1}{p+1}.$

7. Add several of the preceding problems in the ordinary way. Which method is shorter?

52. To multiply a fraction by an integer we extend the definition of multiplication to this case, and we have:

$$3\left(\tfrac{4}{5}\right)=\tfrac{4}{5}+\tfrac{4}{5}+\tfrac{4}{5}=\tfrac{12}{5}.$$

$$c\left(\frac{a}{b}\right)=\frac{a}{b}+\frac{a}{b}+\cdots\text{ to }(c\text{ terms})=\frac{a+a+a\cdots\text{to }(c\,a\text{'s})}{b}=\frac{ca}{b}.$$

Similarly, $(-c)\left(\dfrac{a}{b}\right)=-\dfrac{a}{b}-\dfrac{a}{b}\cdots\text{ to }(c\text{ terms})=-\dfrac{ca}{b}.$

53. That is, *to multiply a fraction by an integer we multiply the numerator by that integer.*

The fraction $\frac{a}{b}$ is the quotient called for in the indicated division $a \div b$, because the product of the divisor b and the asserted quotient $\frac{a}{b}$ is the dividend a, as seen in

$$b\cdot\frac{a}{b}=\frac{ba}{b}=a\quad\text{(by property } B\text{, Sec. 48).}$$

In our present number system the fraction $\frac{a}{b}$ may therefore always be regarded as indicating the division of a by b (provided b is not zero).

54. The operation of division is thus seen to be possible in the new number system for any integral dividend and any integral divisor (except zero).

We shall see, under Division of Fractions, that if either dividend or divisor or both are fractions, the operation is still possible without enlarging the number system further.

55. Multiplication of Fractions. *The product of two or more fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

56. Since an integer or an integral expression may be regarded as a fraction with denominator 1, this definition applies also when one or more of the factors are integral.

57. The Associative, Commutative, and Distributive Laws of Multiplication apply to fractions as well as to integers.

WRITTEN EXERCISES

Multiply:

1. $\frac{5}{7a} \cdot \frac{14}{15b}$.

2. $\frac{x^3}{y^2} \cdot \frac{3xy}{4z^5}$.

3. $\frac{t^2-4}{(1+q)^2} \cdot \frac{q^2-1}{(2+t)^2}$.

4. $\frac{5x}{3y} \cdot \frac{9t^2}{16x^3y} \cdot \frac{4yz}{15t^3x}$.

5. $bc \cdot \frac{b}{c^2} \cdot \frac{ac^4}{2b^3}$.

6. $\frac{-3a}{x^2-9} \cdot \frac{4b}{9x}$.

7. $\left(\frac{3}{a} - \frac{2}{b}\right)^2$.

8. $x^3\left(1 - \frac{1}{x}\right)^2$.

9. $pqr\left(\frac{p}{q} - \frac{q}{r}\right)^2$.

10. $\frac{a(1-a)}{1+2a+a^2} \cdot \frac{1+a}{1-2a+a^2}$.

11. $\frac{1-a^2}{1+2b+b^2} \cdot \frac{1-b^2}{b^2-2ab+a^2}\left(\frac{a}{1-a} - \frac{b}{1-b}\right)$.

12. $\left(\frac{t+1}{t-1} - \frac{t-1}{t+1} - \frac{4t^2}{1-t^2}\right)\frac{t^2+2t+1}{4t}$.

58. Reciprocals. If the product of two numbers is 1, each is called the **reciprocal** of the other.

Thus, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, since $\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ab}=1.$

59. Division of Fractions. Since *divisor* × *quotient* = *dividend,* to divide $\frac{a}{b}$ by $\frac{c}{d}$ means to find a number q such that $\frac{c}{d}\cdot q=\frac{a}{b}.$

Solving the above equation, we have,

$$q=\frac{a}{b}\cdot\frac{d}{c}.$$

That is: *To obtain the quotient, multiply the dividend by the reciprocal of the divisor.*

We have thus seen that in the number system as now extended the four fundamental operations are possible when any or all of the numbers involved are fractions. (Division by zero is always excepted.)

60. Complex Fractions. If one or both of the terms of a fraction are themselves fractions, the given fraction is called a **complex fraction.** Since a fraction indicates division, a complex fraction may be simplified by performing the indicated division of its numerator by its denominator.

WRITTEN EXERCISES

Divide, and express results in lowest terms:

1. $\frac{x}{2}\div\frac{y}{6}.$
2. $\frac{ab}{18c}\div\frac{4b^2}{3ac}.$
3. $\frac{ax^2}{2y^3}\div\frac{bx^3}{ay^2}.$
4. $\frac{9a^2}{4b}\div 2c^2.$
5. $\frac{x^2-1}{7h}\div\frac{x+1}{2m}.$
6. $\frac{-3x}{4y+1}\div\frac{1}{2x-1}.$
7. $\frac{gt^2}{2}\div\frac{ag^3}{8}.$
8. $pv^3+\frac{6p^4}{v^5}.$
9. $\frac{-6}{x^3}\div\frac{-18}{x^7}.$
10. $\frac{\frac{5x}{3y^2}}{\frac{2x}{9m^3}}.$
11. $\frac{\frac{1}{x}}{\frac{3}{x}}.$
12. $\frac{5a}{\frac{3a^2}{2b}}.$
13. $\frac{\frac{4x}{7y^2}}{9x^3}.$
14. $\frac{\frac{1}{t^2}}{t^2}.$
15. $\frac{t^2}{\frac{1}{t^3}}.$

16. $\left(\frac{a^2+b^2}{b}-a\right)\div\left(\frac{1}{b}-\frac{1}{a}\right)$.

17. $\frac{a}{b}+\frac{c}{d}\div\left(1-\frac{ac}{bd}\right)$.

18. $\left(1-\frac{1}{a^3}\right)\div\left(\frac{1}{a^2}-\frac{1}{a^3}\right)$.

Perform the operations indicated:

19. $x-\frac{1}{1+\frac{1}{x}}$. 20. $t+\frac{1}{\frac{1}{t}+1}$. 21. $\left(\frac{p-q}{p+q}+1\right)\div\left(1-\frac{2q}{p+q}\right)$.

FACTORING

61. Factoring. The process of finding two or more factors whose product is a given algebraic expression is called **factoring** the given expression.

In division one factor (the divisor) is given and another factor (the quotient) is to be found such that the product of the two factors is the given expression (the dividend). Division is thus really a variety of factoring, although the name "factoring" is not usually applied to it. As a rule, when an expression is to be factored, none of the factors is specified in advance, and any set of factors is acceptable on the sole condition that their product is the given expression.

62. The type products of special importance were applied in Chapter XII, Part I, and are collected here for reference.

I. $xy+xz=x(y+z)$.

II. $x^2\pm 2xy+y^2=(x\pm y)^2$.

III. $x^2-y^2=(x+y)(x-y)$.

IV. $x^2+(a+b)x+ab=(x+a)(x+b)$.

V. $acx^2+(ad+bc)x+bd=(ax+b)(cx+d)$.

VI. $x^3\pm 3x^2y+3xy^2\pm y^3=(x\pm y)^3$.

VII. $x^3+y^3=(x+y)(x^2-xy+y^2)$.

VIII. $x^3-y^3=(x-y)(x^2+xy+y^2)$.

These formulas apply when the letters are either positive or negative numbers, and a detailed treatment of them was given in Chapter XI, Part I. The following miscellaneous exercises cover all the types.

ORAL EXERCISES

Factor:

1. $ax + ay$.
2. $ax + a$.
3. $ax + a^2$.
4. $abx - ay$.
5. $x^2 + ax$.
6. $x^3 + ax^2$.
7. $a^2x^2 - ax$.
8. $mx + my + m$.
9. $px^2 + p^2x + pq$.
10. $x^3 - x^2 + x$.
11. $x^2y^2 + xy + y$.
12. $(a + b)x - (a + b)y$.
13. $s^2t + st^2 + s^2t^2$.
14. $\frac{1}{2}gt^2 + gt$.
15. $abc - acd + bcd$.
16. $a^2x^2 - b^2y^2$.
17. $4x^2 - 9z^2$.
18. $a^2b^2x^2 - c^2$.
19. $1 - p^2q^2y^2$.
20. $m^2p^2 - s^2t^2$.
21. $(a + b)^2 - (c + d)^2$.
22. $1 - (m + p)^2$.
23. $a^2 + 2abc + b^2c^2$.
24. $a^4x^4 + 2a^2x^2 + 1$.
25. $a^2 + 2a(b + c) + (b + c)^2$.
26. $1 - 2(x + y) + (x + y)^2$.
27. $49 + 14x + x^2$.
28. $9 - 12y^2 + 4y^4$.
29. $16 - 40z + 25z^2$.
30. $a^2b^2 + 2abcd + c^2d^2$.
31. $\frac{1}{4} + t + t^2$.
32. $y^4 + .4y^2 + .04$.
33. $m^2x^2 + 4mx + 4$.
34. $\frac{1}{16}z^4 - \frac{1}{2}z^2 + 1$.
35. $x^2 - 5x + 6$.
36. $x^2 + 7x + 12$.
37. $x^2 - x — 12$.
38. $x^2 + x - 6$.
39. $t^2 - 12x + 35$.
40. $s^2 - 2x - 35$.
41. $12x^2 + 7x + 1$.
42. $15y^2 - 2y - 1$.
43. $6x^2 - 12x + 6$.
44. $3y^2 + 8y + 5$.
45. $a^3 - b^3$.
46. $a^3 + b^3$.
47. $8 - a^3b^3$.
48. $64 - x^3y^3$.
49. $8x^3 - 12x^2 + 6x - 1$.
50. $27y^3 + 27y^2 + 9y + 1$.

WRITTEN EXERCISES

Factor:

1. $x^2 - 49$.
2. $t^2 - 7t + 6$.
3. $5y^3 - 3y^2 + y$.
4. $4v^2 - 9u^2$.

5. $1+10a+25a^2$.
6. $a^3+3a^2b+3ab^2+b^3$.
7. $(x+1)^2+2(x+1)+1$.
8. $1-\frac{2}{a}+\frac{1}{a^2}$.
9. $x^2-2x-15$.
10. a^2b^2-1.
11. $x^4+6x^2y+9y^2$.
12. v^3-t^4.
13. $1-x^3y^3$.
14. $a^4-4a^2b^2$.
15. $(x+y)^4-1$.
16. $a^3b^3-c^3d^3$.
17. $x^4-2x^2y^2+y^4$.
18. $(m+p)^3-1$.
19. $(x+y)^2-(p+q)^2$.
20. $(a+b)^3+x^3$.
21. $x^3+\frac{3}{2}x^2+\frac{3}{4}x+\frac{1}{8}$.
22. $a^3-1\frac{1}{2}a^2+\frac{3}{4}a-\frac{1}{8}$.
23. $x^2+(a+b)x+ab$.
24. $y^2+(ac-bd)y-abcd$.
25. $acx^2+(cb+ad)x+bd$.
26. $.125x^3+.75x^2+1.5x+1$.
27. $\frac{x^3}{8}-\frac{3x^2}{4}+\frac{3x}{2}-1$.
28. $8a^3+12a^2t+6at^2+t^3$.
29. g^4-16.
30. $a^2y^2-11ay+24$.
31. $4h^2-12ht+9t^2$.
32. $1-m^3$.
33. $(2a+b)^2-(3a-2b)^2$.
34. $8a^3-32a^5$.
35. $10m^2x-60mx+90x$.
36. $m^4p^4+4s^4$.
37. x^4+3x^2-28.
38. s^3-64.
39. $56x^2-68x+20$.
40. $24x^2+x-10$.
41. $15x^2-34x-16$.
42. $a^2z^2-7az+12$.
43. y^4-13y^2+42.
44. $(a+1)^2-(a-1)^2$.
45. $x^2-\frac{1}{2}x-\frac{1}{2}$.

Calculate:

46. 27^2-25^2. 47. 387^2-377^2. 48. 26^3-25^3.

General Methods

63. The General Trinomial. The factors of the general trinomial mx^2+px+q have the form $(ax+b)(cx+d)$, and they may be found by reducing the general form to the type x^2+bx+c. To do this, multiply and divide the expression by m (the coefficient of x^2) and put $mx=y$. The method will be made sufficiently clear by an example.

EXAMPLE

Factor: $6x^2 + 19x + 10.$

Multiplying and dividing the given trinomial by 6, we obtain

$$6x^2 + 19x + 10 = \tfrac{1}{6}(36x^2 + 6 \cdot 19x + 60).$$

Put $\quad 6x = y. \quad (1)$

Then, $\quad 36x^2 + 6 \cdot 19x + 60 = y^2 + 19y + 60 \quad (2)$

$= (y + 4)(y + 15)$ Sec. 163, Pt. I. (3)

Replacing y by $6x$, $\quad = (6x + 4)(6x + 15) \quad (4)$

$= 2(3x + 2)3(2x + 5)$

$= 6(3x + 2)(2x + 5). \quad (5)$

And finally, $\quad 6x^2 + 19x + 10 = \tfrac{1}{6} \cdot 6(3x + 2)(2x + 5)$

$= (3x + 2)(2x + 5). \quad (6)$

TEST by multiplication.

WRITTEN EXERCISES

Factor:

1. $3x^2 + 7x + 2.$
2. $3x^2 - 5x - 2.$
3. $4x^2 + 13x - 12.$
4. $6x^2 - 22x + 20.$
5. $3a^2 + a - 10.$
6. $4m^2 + 9m - 9.$
7. $4p^2 + 8p + 3.$
8. $9x^2 - 3x - 2.$
9. $5x^2 - 9x - 2.$
10. $6a^2 - 8a - 8.$
11. $25y^2 - 50y - 24.$
12. $10x^2 + 9xy - 9y^2.$
13. $15p^2 - 18p - 24.$
14. $12s^2 - 10s - 12.$
15. $6y^2 + 35y - 6.$
16. $28z^2 - 44z + 16.$
17. $7p^2 + 22p + 3.$
18. $15z^2 + z - 6.$
19. $3a^2 + 14ab - 5b^2.$
20. $15x^2 + 2xy - 8y^2.$
21. $14m^2 - mp - 3p^2.$
22. $10s^2 + 9st - 9t^2.$
23. $12x^4 - 47x^2 - 17.$
24. $40x^{2a} - 2x^a - 2.$
25. $3a^2 - 8a - 115.$
26. $6a^2b^2 - ab^3 - 12b^4.$
27. $25x^2 + 20x + 4.$
28. $4z^2 + 4az - 15a^2.$
29. $3a^2 + ab - 2b^2.$
30. $12y^2 + 19y - 21.$
31. $6m^2 + mn - 2n^2.$
32. $96a^2 - 4ab - 15b^2.$

64. Expressions that can be made the Difference of Two Squares.

1. Expressions of the form $a^{4n}+4\,b^{4n}$ may be factored by adding and subtracting $4\,a^{2n}b^{2n}$. Thus,

$$\begin{aligned}a^{4n}+4\,a^{2n}b^{2n}+4\,b^{4n}-4\,a^{2n}b^{2n}&=(a^{2n}+2\,b^{2n})^2-(2\,a^nb^n)^2\\&=(a^{2n}+2\,b^{2n}-2\,a^nb^n)(a^{2n}+b^{2n}+2\,a^nb^n).\end{aligned}$$

EXAMPLE

$$\begin{aligned}x^8+4\,y^8&=x^8+4\,x^4y^4+4\,y^8-4\,x^4y^4\\&=(x^4+2\,y^4-2\,x^2y^2)(x^4+2\,y^4+2\,x^2y^2).\end{aligned}$$

2. Expressions of the form $a^{4n}\pm pa^{2n}b^{2n}+b^{4n}$ may be factored if a number can be found such that when added to p the result is 2.

EXAMPLE

Factor $a^4-6\,a^2b^2+b^4$.

Adding and subtracting $4\,a^2b^2$,

$$\begin{aligned}a^4-6\,a^2b^2+b^4&=a^4-2\,a^2b^2+b^4-4\,a^2b^2.\\&=(a^2-b^2)^2-4\,a^2b^2.\\&=(a^2-b^2-2\,ab)(a^2-b^2+2\,ab).\end{aligned}$$

3. Some other expressions may be made the difference of two squares.

EXAMPLE

Factor $16\,m^4+36\,m^2p^2+25\,p^4$.

Since twice the product of the square roots of the end terms is $40\,m^2p^2$, the expression can be made the difference of two squares by adding and subtracting $4\,m^2p^2$. Thus,

$$\begin{aligned}16\,m^4+36\,m^2p^2+25\,p^4&=16\,m^4+40\,m^2p^2+25\,p^4-4\,m^2p^2\\&=(4\,m^2+5\,p^2)^2-(2\,mp)^2\\&=(4\,m^2+5\,p^2-2\,mp)(4\,m^2+5\,p^2+2\,mp).\end{aligned}$$

WRITTEN EXERCISES

Factor:

1. $p^4+4\,q^4$.
2. m^4+4.
3. $4\,x^4+1$.
4. $4\,y^8+1$.
5. $y^{12}+4$.
6. $4\,m^8+y^8$.
7. a^4+a^2+1.
8. $a^4+4\,a^2+16$.
9. $a^4+a^2b^2+b^4$.

10. $64p^4+1.$ **11.** $m^8+64.$ **12.** $4p^{8m}+q^{4m}.$

13. $9x^4-10x^2+1.$ **15.** $4c^4+3c^2d^2+9d^4.$

14. $16y^4+4y^2+1.$ **16.** $36x^4-84x^2y^2+16y^4.$

65. The General Binomial. The factors of the general binomial $x^n \pm y^n$ depend upon the nature of n. The following table shows for what values of n the division is exact in $(x^n \pm y^n) \div (x \pm y)$:

a. $(x^n+y^n) \div (x+y)$, if n is odd.
b. $(x^n+y^n) \div (x-y)$, not for any value of n.
c. $(x^n-y^n) \div (x+y)$, if n is even.
d. $(x^n-y^n) \div (x-y)$, for every integral value of n.

ORAL EXERCISES

Name a factor of each of the following:

1. $x^5-y^5.$ **3.** $1-x^6.$ **5.** $x^{10}-1.$ **7.** $a^{14}-b^{14}.$

2. $x^7+a^7.$ **4.** $x^9+a^9.$ **6.** $x^{13}-y^{13}.$ **8.** $32x^5-1.$

66. The factor of the first degree in $x^n \pm y^n$ having been found by inspection, the other is a regular series of powers that can be written directly (Sec. 39); it has alternate signs in the cases (*a*) and (*c*), and plus signs in case (*d*).

For example:

$$x^7+a^7=(x+a)(x^6-ax^5+a^2x^4-a^3x^3+a^4x^2-a^5x+a^6).$$

$$\begin{aligned}32a^5-1&=(2a)^5-1\\&=(2a-1)[(2a)^4+(2a)^3\cdot 1+(2a)^2\cdot 1^2+2a\cdot 1^3+1^4]\\&=(2a-1)(16a^4+8a^3+4a^2+2a+1).\end{aligned}$$

$x^{10}-r^{10}$ has both $x-r$ and $x+r$ as factors, but it is better first to factor as a difference of two squares; then apply (*d*) and (*a*).

WRITTEN EXERCISES

Factor:

1. $32a^5-b^5.$ **5.** $a^7-b^7.$ **9.** $(x-3y)^5+1.$

2. $128a^7+1.$ **6.** $1-m^{10}.$ **10.** x^8-y^8 (four factors).

3. $32-243t^5.$ **7.** $x^6-a^6.$ **11.** $(a+b)^5-(a-b)^5.$

4. $(c-d)^7+1.$ **8.** $1-(a+b)^4.$ **12.** $(x-y)^4-(x+y)^4.$

67. Factor Theorem. *If any polynomial in* x *assumes the value zero when* a *is substituted for* x, *then* x − a *is a factor of the polynomial.*

For example, substitute 2 for x in the polynomial:

$$x^4 - 3x^3 + 7x^2 - 2x - 16.$$

Then, $2^4 - 3 \cdot 2^3 + 7 \cdot 2^2 - 2 \cdot 2 - 16 = 16 - 24 + 28 - 4 - 16 = 0.$

Suppose the polynomial to be divided by $x - 2$, and denote the quotient by Q and the remainder by R, the latter being numerical.

Then, $x^4 - 3x^3 + 7x^2 - 2x - 16 = (x - 2)Q + R.$

In this equation substitute 2 for x; the left number becomes zero as just seen; $2 - 2$ is also 0, and 0 times Q is 0; hence the result is:

$$0 = 0 + R, \text{ or } R = 0.$$

Consequently, $x^4 - 3x^3 + 7x^2 - 2x - 16 = (x - 2)Q$, or $x - 2$ is a factor of $x^4 - 3x^3 + 7x^2 - 2x - 16$. Thus we may know *without actual division* that $x - 2$ is a factor of the polynomial.

In general: Let $P(x)$ denote the given polynomial.

Suppose $P(x)$ to be divided by $x - a$. There will be a certain quotient, call it $Q(x)$, and a remainder, R. This remainder will not involve x, otherwise the division could be continued.

We have then: $P(x) = (x - a)Q(x) + R.$ (1)

Put a for x, and denote the resulting value of $P(x)$ by $P(a)$, and of $Q(x)$ by $Q(a)$:

Then, $P(a) = (a - a)(Q)(a) + R.$ (2)
But, $(a - a)(Q)a = 0,$
hence, if $P(a) = 0,$
then, $R = 0,$
and, $P(x) = (x - a)Q(x).$

That is, $x - a$ is a factor of $P(x)$, if the expression is 0 when $x = a$.

WRITTEN EXERCISES

By use of the Factor Theorem test each expression and find if the binomial at the right is a factor of it:

Expression	Binomial
1. $x^2 - 4x + 4.$	$x - 2.$
2. $x^2 - 5x + 6.$	$x - 3.$

3. x^2-x-6.	$x-3$.
4. $3x^2-x-2$.	$x-1$.
5. $3y^2-20y+25$.	$y-5$.
6. x^3-3x^2+3x-1.	$x-1$.
7. $p^3-4p^2-4p+16$.	$p-4$.
8. m^3-2m^2+m-2.	$m-5$.

68. *If the polynomial becomes* 0 *when* − a *is put for* x, *then* x −(− a), or x + a *is a factor.*

Example: $x+2$ is a factor of x^3+x^2-2x because

$$(-2)^3+(-2)^2-2(-2)=-8+4+4=0.$$

WRITTEN EXERCISES

By use of the Factor Theorem, prove that each polynomial has the factor named:

POLYNOMIAL	FACTOR
1. $x^3+12x^2+31x-20$.	$x+5$.
2. $(x-a)^2+(x-b)^2-(a-b)^2$.	$x-b$.
3. x^3+2x^2+3x+2.	$x+1$.
4. $x^3-2x^2+\frac{3x}{m}-\left(m^3-2m^2+\frac{3m}{x}\right)$.	$x-m$.
5. $x^{3n}+2x^{2n}+3x^n+2$.	x^n+1.
6. $x^3+ax^2-a^2x^2-a^5$.	$x-a^2$.
7. $9x^5+(3a^2-12a)x^4-4a^3x^3+3a^2x+a^4$.	$x+\frac{a^2}{3}$.

In each polynomial, substitute the values given for x, and use the Factor Theorem to find the possible factors:

POLYNOMIAL	VALUES FOR x
8. $x^4+3x^3+4x^2-12x-32$.	1, 2, − 2.
9. $x^4-7x^3+14x^2+x-21$.	1, − 1, 2, 3.
10. $2x^4+5x^3-41x^2-64x+80$.	4, − 4, 5, − 5.

11. Apply the Factor Theorem to $(a^n \pm b^n) \div (a \pm b)$ and prove the results of Secs. 65 and 66.

REVIEW

WRITTEN EXERCISES

Perform the indicated operations, expressing fractional results in lowest terms:

1. $2a-3b+4a+11c-2d+6-8a-9b+3c-4-5d+2a-b$.

2. $x^2y+3xy^2+4x^3-2xy-3x^2y+2x^2y^2-4xy+8xy^2-7y^2-3x^2y^2$.

3. $5x-7y-(3x+4y-2)+3+(8x-7)-(2x-8y+13)+8(2x-1)$.

4. $4x-\{2c-(3x+2y+5c)\}$.

5. $5ac-\{4b+2c[3a-5b-(6a-2b-\overline{7c+3})]\}$.

If $L=4x+2y-3$, $M=x-9y+1$, $R=y-5x$, find:

6. $L+M+R$. 8. $LM-3R$. 10. $(L+M)^2$.

7. $3L-2M+R$. 9. L^2-M^2. 11. $3L^2-5MR$.

If $X=7a+2b$, $Y=2a-9b+3$, $Z=4b-7a-3$, find:

12. $X-(3Y-2Z)$. 13. $2Y-[Z+3(4X-8Y)]$.

14. X^2-16Y^2. 15. Y^2+4YZ. 16. XYZ.

17. $\dfrac{a+5}{a-5}-\dfrac{a^2+75}{a^2-25}+\dfrac{a-5}{a+5}$.

18. $\dfrac{6m+4p}{15m}-\dfrac{3m^2+4p^2}{15mp}+\dfrac{m-2p}{5p}$.

19. $1-\dfrac{9+a}{1+a^2}-\dfrac{a-3}{a+1}$. 20. $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)(x+y)$.

21. $\dfrac{2}{t^2+9t+20}+\dfrac{1}{t^2+7t+12}-\dfrac{3}{t^2+11t+28}$.

22. $c^4\left(\dfrac{x^2}{c^3}-\dfrac{3x}{c^2}+\dfrac{14}{c}\right)$. 23. $\dfrac{1}{3a^2}\left(\dfrac{6a^3+9a^2}{4a+2}\right)$.

24. $(b+3)^3-(b-2)^3$.

25. $(x^n+3x^{n-1}+5x^{n-2})(x^2+4x)$.

26. $\left(\dfrac{10x^3}{3m^2}-\dfrac{5x^3}{9m^2}-\dfrac{5x^2}{m^4}\right)\div\dfrac{15x}{m^2}$.

27. $\left(\frac{2x^2}{y^2}-\frac{5y^2}{x^2}+3\right)\div\left(\frac{x}{y}-\frac{y}{x}\right).$

28. $\frac{35+12g+g^2}{15+8g+g^2}\div\frac{56+15g+g^2}{24+11g+g^2}.$

29. $\dfrac{\frac{1}{a}-\frac{3}{a^2}}{\frac{2}{a^3}-\frac{3}{a}}.$ 30. $\dfrac{\frac{q+1}{q-1}+\frac{q-1}{q+1}}{\frac{q+1}{q-1}-\frac{q-1}{q+1}}.$

31. $\left(\frac{3a}{b}-\frac{11b}{3a}\right)\left(\frac{3a}{b}-\frac{b}{3a}\right)-\left(\frac{3a}{b}-\frac{2b}{3a}\right)^2.$

32. $\frac{2}{x^2+4x+3}+\frac{1}{x^2+3x+2}-\frac{3}{x^2+5x+6}.$

33. $\frac{x^2y}{a^3+1}\div\frac{2x^3}{a+1}.$

34. $\frac{4a+7}{a^2-16x+64}\div\frac{16a^2+56a+49}{a^2-64}.$

Divide by detached coefficients:

35. $16-32x+24x^2-8x^3+x^4\div(2-x).$

36. $16m^4-32m^3+24m^2-8m+1\div(2m-1).$

Factor by the Factor Theorem:

37. $x^3+x^2-x-1.$

38. $4x^4+x^3-5.$

39. $a^3-ab^2+a^2-ab.$ (Substitute b for a.)

SUMMARY

1. State the laws that govern the processes of addition for algebraic numbers. Secs. 3, 9.

2. State the laws that govern the processes of multiplication for algebraic numbers. Secs. 26, 32.

3. State and illustrate two extensions of the number system of algebra. Secs. 16, 45.

4. Name eight type expressions used in factoring. Sec. 62.

5. State the Factor Theorem. Sec. 67.

HISTORICAL NOTE

We have seen that algebra embraces the enlargement of the number field of arithmetic to include negative number and the extension of the basic processes to those numbers. Thus, the equation $x + 3 = 1$, which Ahmes could not solve and which was regarded as negligible until the sixteenth century, is no more exceptional than $x + 1 = 3$, since algebra defines $1 - 3$ to be -2. In a similar way the field of number was enlarged to include positive and negative fractions, for the equation $3x = -2$ could not be solved until $\frac{-2}{3}$ was understood. In arithmetic the processes $b - a$ and $\frac{b}{a}$ are limited to positive numbers, but in algebra each of these has a meaning for all values of b and a, positive or negative. This process of generalizing was first explained by the English algebraist, George Peacock (1830), and called by him the *principle of permanence.* That is, in order that any number or symbol may be made a part of algebra, it must conform to the basic laws governing the processes, namely, The Associative Law, The Commutative Law, and The Distributive Law.

Sir William Rowan Hamilton

The famous scientist, Sir William Rowan Hamilton (1840), regarded these laws as distinguishing algebraic number from other numbers. In doing so, he discovered numbers which do not obey the Commutative Law of Multiplication, and to these numbers he gave the name Quaternions. Their study has since become a new branch of mathematics.

Hamilton was of Scotch parentage, but Ireland shares his fame, because he was born and educated at Dublin. Like Tartaglia, he received instruction at home when a boy, and showed exceptional ability at an early age. When only thirteen he could read a dozen languages, at eighteen he had mastered Newton's *Principia*, and shortly became professor of Astronomy in Trinity College, Dublin. Hamilton did much for mechanics and astronomy, but his greatest achievement in mathematics was the discovery of Quaternions.

CHAPTER II

EQUATIONS

EQUATIONS WITH ONE UNKNOWN

69. Two algebraic expressions are **equal** when they represent the same number.

70. If two numbers are equal, the numbers are equal which result from:

1. *Adding the same number to each.*
2. *Multiplying each by the same number.*

Subtraction and division are here included as varieties of addition and multiplication.

71. The equality of two expressions is indicated by the symbol, =, called "the sign of equality."

72. Two equal expressions connected by the sign of equality form an **equation.**

73. Such values of the letters as make two expressions equal are said to **satisfy** the equation between these expressions.

74. Equations that are satisfied by any set of values whatsoever for the letters involved are called **identities.**

75. Equations that are satisfied by particular values only are called **conditional** equations, or, when there is no danger of confusion, simply equations.

76. The numbers that satisfy an equation are called the **roots** of the equation.

77. **To solve** an equation is to find its roots.

78. The letters whose values are regarded as unknown are called the **unknowns.**

79. The **degree** of an equation is stated with respect to its unknowns. It is the highest degree to which the unknowns occur in any term in the equation. Unless otherwise stated, all the unknowns are considered.

80. An equation of the first degree is called a **linear** equation.

81. An equation of the second degree is called a **quadratic** equation.

82. An equation of the third or higher degree is called a **higher** equation.

83. In order to state the degree of an equation its terms must be united as much as possible.

84. Terms not involving the unknowns are called **absolute** terms.

85. **Equivalent Equations.** If two equations have the same roots, the equations are said to be equivalent. If two equations have together the same roots as a third equation, the two equations together are said to be equivalent to the third.

86. **The Linear Form, $ax+b$.** Every polynomial of the first degree can be put into the form $ax+b$. That is, by rearranging the terms suitably, it can be written as the product of x by a number not involving x, plus an absolute term. Hence, the form $ax+b$ is called a general form for all polynomials of the first degree in x.

87. Every equation of the first degree in one unknown can be put into the form:

$$ax+b=0.$$

Consequently this is called *a general equation of the first degree in one unknown.*

88. **General Solution.** From the equation $ax+b=0, \quad (1)$

we have $ax=-b, \quad (2)$

and hence, $x=\frac{-b}{a}. \quad (3)$

89. $-\frac{b}{a}$ is the general form of the root of the equation of the first degree. There is always one root, and only one.

The advantage of a general solution like this is that it leads to a formula which is applicable to all equations of the given form.

In words: *When the equation has been put into the form* $ax + b = 0$ *the root is the negative of the absolute term divided by the coefficient of* x.

Test. The correctness of a root is tested by substituting it in the *original equation.*

If substituted in any later equation, the work leading to that equation is not covered by the test.

Results for problems expressed in words should be tested by substitution in the *conditions of the problem.*

If tested by substitution in the equation only, the correctness of the solution is tested, but the setting up of the equation is not tested. Negative results that may occur in such problems are always correct as solutions of the equations, but they are admissible as results in the concrete problem only when the unknown quantity is such that a unit of the unknown quantity is offset by a unit of its opposite.

For example, if the unknown measures distance forward, a negative result means that a corresponding distance backward satisfies the conditions of the problem. But, if the unknown is a number of men, a negative result is inadmissible, since no opposite interpretation is possible.

ORAL EXERCISES

Solve for x:

1. $3x = 15.$
2. $2x = 11.$
3. $4\frac{1}{2}x = 9.$
4. $x - 6 = 10.$
5. $x + 6 = 12.$
6. $2x + 1 = 13.$

Solve for t:

7. $6t = 36.$
8. $t - 5 = 20.$
9. $2t + 5 = 25.$
10. $3t - 8 = 22.$
11. $at = ab.$
12. $at + b = c.$

Solve for y:

13. $3y-1=2$.

14. $2y-1=7$.

15. $5y+5=35$.

16. $by=bc$.

17. $by=b+c$.

18. $ay-b=c$.

Solve for a:

19. $6a=18$.

20. $2\frac{1}{2}a=10$.

21. $a+1=13$.

22. $3a+1=13$.

23. $2a-5=15$.

24. $5a-b=c$.

WRITTEN EXERCISES

Solve for x:

1. $x-75=136$.

2. $2x-15=45$.

3. $3x+6=48$.

4. $5x-10=55$.

5. $1.5x-5=70$.

6. $3.5x-5=100$.

7. $2.1x-41=400$.

8. $1.3x+.1=1.79$.

9. $.25x+.50=3.25$.

10. $.11x+.11=1.32$.

11. $325x+60=400$.

12. $125x+5=10$.

13. $8x+625=105$.

14. $ax+bx=c$.

15. $abx+ax=ab$.

16. $cx+dx=c+d$.

17. $mx+px=p+q$.

18. $lx+mx=l-m$.

19. $ax+bx=2(a+b)$.

20. $2cx+dx=1$.

Solve and test:

21. $2(x-1)+3(2x+5)=0$.

22. $\frac{4(x+2)}{5}+3=2x+1$.

23. $\frac{x-1}{2}+\frac{x-1}{3}+\frac{x-1}{4}=x$.

24. $\frac{3y+1}{4y-2}=\frac{1-3y}{4-4y}$.

25. $\frac{1+3p}{p}=\frac{17-5p}{p}$.

Solve for c:

26. $5c + 3a = ac.$

27. $\frac{4c-1}{3m} = \frac{2c+5}{6m}.$

28. $v = ct + \frac{gt^2}{2}.$

29. $(a+c)(a-c) = -(c+a)^2.$

30. $\frac{3c}{c-3} - \frac{c-2}{c-4} = \frac{(c-2)(2c-7)}{c^2-7c+12}.$

31. $\frac{L}{1+ct} = l.$

32. An inheritance of $2000 is to be divided between two heirs, A and B, so that B receives $100 less than twice what A receives. How much does each receive?

33. Twenty steps, each of a given height, are required to build a certain staircase. If each step is made 2 inches higher, 16 steps are required. Find the vertical height of the staircase.

34. In a certain hotel the large dining room seats three times as many persons as the small dining room. When the large dining room is $\frac{3}{8}$ full and the small dining room $\frac{1}{2}$ full, there are 100 persons in both together. How many does each room seat?

35. Tickets of admission to a certain lecture are sold at two prices, one 25 cents more than the other. When 100 tickets at the lower price and 60 at the higher price are sold, the total receipts are $95. Find the two prices.

36. Originally, $\frac{10}{11}$ of the area of Alabama was forest land. $\frac{1}{8}$ of this land has been cleared, and now 20 million acres are forest land. Find the area of Alabama in million acres.

37. In a recent year the railroads of the United States owned 70,000 cattle cars. Some of these were single-decked, and others double-decked. There were 44,000 more of the former than of the latter. Find how many cars there were of each kind.

38. The average number of sheep carried per deck is 45 larger than the average number of calves. If a double-decked car has the average number of calves on the lower deck and of sheep on the upper deck, it contains 195 animals. Find the number of sheep and of calves.

39. The average number of inhabitants per square mile for Indiana is $\frac{7}{4}$ of that for Iowa, and that for Ohio is 32 greater than that for Indiana, and 62 greater than that for Iowa. Find the number for each state.

40. Lead weighs $\frac{63}{15}$ times as much as an equal volume of aluminium. A certain statuette of aluminium stands on a base of lead. The volume of the base is twice that of the statuette, and the whole weighs 282 oz. Find the weight of the statuette and of the base.

41. A man inherits $10,000. He invests some of it in bonds bearing $3\frac{1}{2}$ % interest, the rest in mortgages bearing $5\frac{1}{2}$ % interest per annum. His entire annual income from these investments is $ 510. Find the amount of each investment.

42. A pile of boards consists of inch boards and half-inch boards. There are 80 boards and the pile is 58 in. high. How many boards of each thickness are there?

43. A hardware dealer sold a furnace for $ 180 at a gain of 20 %. What did the furnace cost him?

44. A merchant sold a damaged carpet for $ 42.50 at a loss of 15 %. What did the carpet cost him?

45. A collector remitted $ 475 after deducting from the amount collected a fee of 5 %. How many dollars did he collect?

46. The amount of a certain principal at 4 % simple interest for 1 yr. was $ 416. What was the principal?

47. Pythagoras being asked the time of day, replied: "There remains of the day (from 6 A.M. to 6 P.M.) twice the number of hours already passed." What time was it?

48. The three Graces, carrying 4 apples each, met the 9 Muses; they gave each Muse the same number of apples; then the Muses and Graces had equal shares. How many had each?

49. A fruit vender gave a boy 4 dozen oranges to sell and agreed to pay him $\frac{3}{4}$ ¢ for each orange sold, but demanded a payment of 2 ¢ for each orange eaten; the boy disposed of all the oranges and received 25 ¢. How many oranges did he eat?

50. A robber in escaping from a castle met a guard whom he bribed with $\frac{1}{5}$ of his plunder; at the next gate he bribed another guard with $\frac{1}{4}$ of the plunder remaining; at the third gate he bribed another guard with $\frac{1}{3}$ of the plunder remaining; the robber then escaped with 2000 ducats. How many ducats did he steal?

51. A servant agreed to work for £10 a year and his livery. At the end of 7 months his lord discharged him, giving him the livery only. How many pounds was the livery worth?

SPECIAL PROBLEMS

90. The solution of many problems is made easier by a special plan of work.

EXAMPLE

How much water must be added to a 20 % solution of ammonia to make a 10 % solution?

PLAN. 1. Consider an arbitrary quantity of the given mixture; for example, 1 gallon.

2. Let x be the number of gallons added; then the two quantities are:

1st Quantity	2d Quantity
1 gal.	$(1 + x)$ gal.

3. Decide which substance (the ammonia in this problem) has not changed in quantity; state the amount in each solution. Thus,

Ammonia in 1st Solution	Ammonia in 2d Solution
20 % of 1 gal.	10 % of $(1 + x)$ gal.

4. $\therefore$ the equation is 20 % of 1 = 10 % of $(1 + x)$, or $.2 = .1\,(1 + x)$.

5. Therefore, $x = 1$, and 1 gallon of water must be added for each gallon of the original solution.

WRITTEN EXERCISES

Solve the following, using the tabular plan given above:

1. How much water must be added to a 95 % solution of alcohol to make an 80 % solution?

2. A certain paint consists of equal parts of oil and pigment. How much oil must be added to a gallon of this paint to make a paint $\frac{2}{3}$ of which is oil?

3. How much potash must be added to a 10 % solution to make a 20 % solution?

4. Spirits of camphor is camphor gum dissolved in alcohol. How many ounces of camphor gum must be added per ounce to a 5 % solution in order to make an 8 % solution?

91. Problems involving the rates of two moving bodies have received much attention in mathematics.

For example, a courier, or messenger, leaves the rear of an army, 5 miles long, to deliver a dispatch to the officer at the front, and rides at the rate of 10 miles per hour. 10 minutes later he is followed by another messenger riding at the rate of 15 miles per hour. If the army is not moving, where will the second messenger overtake the first?

The following are further examples commonly known as "clock" problems and "planet" problems.

EXAMPLE

At what time between 3 and 4 o'clock are the hands of a clock together?

SOLUTION. 1. The minute hand moves 1 minute space per minute.

2. The hour hand moves $\frac{1}{12}$ of a minute space per minute.

3. Let $x =$ the number of minutes *after* 3 *o'clock* when the hands are together. Then, x is the number of spaces moved by the minute hand and $\frac{x}{12}$ is the number of spaces moved by the hour hand.

4. When they are together the hour hand is $15 + \frac{x}{12}$ minute spaces from XII, and the minute hand is x spaces.

5. Hence, $x = 15 + \frac{x}{12}$, and $x = 16\frac{4}{11}$, the number of minutes past 3 o'clock.

WRITTEN EXERCISES

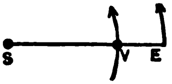
1. How many minute spaces must the minute hand gain on the hour hand from the time they meet until they lie opposite to each other in the same straight line? At what time are the hands of a clock opposite to each other for the first time after 12 o'clock?

2. At what time between 7 and 8 o'clock are the hands of a clock opposite each other?

3. At how many different times, and when, are the hands of a clock at right angles between 4 and 5 o'clock?

4. A and B enter a race together; at the end of 5 minutes A is 900 yd. from the starting line and 75 yd. ahead of B; at this point he falls, and though he renews the race, his rate is 20 yd. a minute less for the rest of the course; he crosses the line ½ minute after B. How long did the race last?

5. In astronomy it is important to know when planets are in line between the earth and the sun. This is called *conjunction*. Taking the earth's time of revolution about the sun as 365 days and that of Venus as 225 days, how long after one conjunction of Venus until the next one occurs?



SUGGESTION. The problem is quite analogous to that of the hands of a watch. For the purposes of this problem we suppose all the planets to move in the same plane and in circular paths (orbits) in the same direction of revolution about the sun as a center.

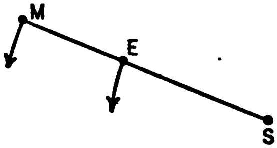
1. Let $x =$ the number of days.
2. Venus will have made $\frac{x}{225}$ revolutions.
3. The earth will have made $\frac{x}{365}$ revolutions.
4. But to be in conjunction, Venus (which goes faster) must have made one more revolution than the earth.

$$\text{Hence,}\quad \frac{x}{225} = \frac{x}{365} + 1,$$

$$\text{and}\qquad x = 586\tfrac{17}{28}.$$

6. Taking 88 days as Mercury's time of revolution about the sun, how long from one conjunction of Mercury to the next?

7. When the earth is between a planet and the sun, in the same line with them, the planet is said to be in opposition to the sun. Taking 687 days as Mars' time of revolution about the sun, how long is it from one opposition of Mars to the next?



8. Taking 4307 days as Jupiter's time of revolution about the sun, how long is it from one opposition of Jupiter to the next?

9. Answer the same question for Saturn, whose time of revolution is 28.5 yr.

10. Also for Neptune, whose time of revolution is 165.5 yr.

NOTE. Those who have studied geometry may take up here some of the problems based upon geometric properties found in Chapter X.

EQUATIONS WITH TWO UNKNOWNS

92. **Systems of Equations.** Two or more equations considered together are called a **system of equations.**

93. **Simultaneous Equations.** Two or more equations are said to be **simultaneous** when all of them are satisfied by the same values of the unknowns.

94. All systems of two independent simultaneous equations of the first degree in two unknowns can be solved by the **method of addition and subtraction,** which consists in multiplying one or both of the given equations by such numbers that the coefficients of one of the unknowns become numerically equal. Then by addition or subtraction this unknown is eliminated, and the solution is reduced to that of a single equation.

95. Occasionally the **method of substitution** is useful. This consists in expressing one unknown in terms of the other by means of one equation and substituting this value in the other equation, thus eliminating one of the unknowns.

This may be the shorter method when an unknown in either equation has the coefficient 0, $+1$, or -1.

General Solution. A general form for two equations of the first degree is

$$ax + by = e, \tag{1}$$
$$cx + dy = f. \tag{2}$$

From these it is possible (without knowing the values of a, b, c, d, e, f) to find a *general form* for the solution, namely:

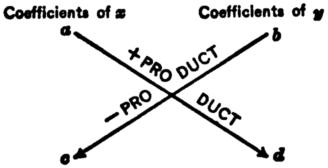
$$x = \frac{de - bf}{ad - bc}, \qquad y = \frac{af - ce}{ad - bc}. \tag{3}$$

These are the *formulas* for the roots of any two independent linear simultaneous equations with two unknowns.

96. Relation of the Roots to the Constants in the Equations.

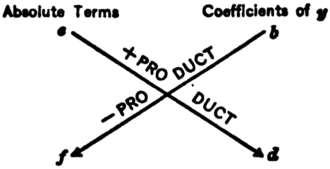
1. The denominator is the same in each result and is made up from the coefficients as follows:

$$\begin{cases} ax + by = e. \\ \\ cx + dy = f. \end{cases}$$



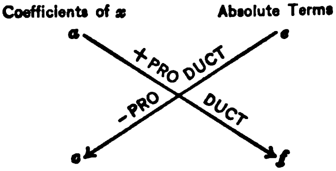
2. The numerator of the value of x is made up thus:

$$\begin{cases} ax + by = e. \\ \\ cx + dy = f. \end{cases}$$



3. The numerator of the value of y is made up thus:

$$\begin{cases} ax + by = e. \\ \\ cx + dy = f. \end{cases}$$



The following examples will illustrate the use of these formulas in solving equations:

1. Solve: $\begin{cases} 2x - 3y = 4, & (1) \\ 4x + 2y = 1. & (2) \end{cases}$

$$x = \frac{\begin{matrix} 4 & & -3 \\ & \times & \\ 1 & & +2 \end{matrix}}{\begin{matrix} 2 & & -3 \\ & \times & \\ 4 & & 2 \end{matrix}} = \frac{4 \cdot 2 - 1(-3)}{2 \cdot 2 - (-3) \cdot 4} = \frac{8+3}{4+12} = \frac{11}{16}. \qquad (3)$$

$$y = \frac{\begin{matrix} 2 & & 4 \\ & \times & \\ 4 & & 1 \end{matrix}}{\begin{matrix} 2 & & -3 \\ & \times & \\ 4 & & 2 \end{matrix}} = \frac{2 \cdot 1 - 4 \cdot 4}{2 \cdot 2 - 4(-3)} = \frac{2-16}{16} = \frac{-14}{16} = \frac{-7}{8}. \qquad (4)$$

TEST. $\frac{2 \cdot 11}{16} - \frac{3 \cdot -7}{8} = \frac{64}{16} = 4;\ \frac{4 \cdot 11}{16} + \frac{2 \cdot -7}{8} = \frac{16}{16} = 1.$

97. The above form of expressing cross products is derived from the **Determinant Notation**, and while it is not necessary to know **Determinants** in order to solve such simultaneous equations by inspection, it is well to know the basis of the method.

The symbol $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is defined to mean $ad - bc$, and is called a Determinant of the *second order*.

EXAMPLES

$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 2 \cdot 5 - 3 \cdot 4 = -2$, and $\begin{vmatrix} 2 & -1 \\ -3 & 5 \end{vmatrix} = 2 \cdot 5 - (-3 \cdot -1) = 13.$

ORAL EXERCISES

Find the value of:

1. $\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}.$
2. $\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix}.$
3. $\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix}.$
4. $\begin{vmatrix} \frac{1}{2} & 3 \\ 6 & 4 \end{vmatrix}.$
5. $\begin{vmatrix} \frac{1}{2} & \frac{1}{4} \\ 8 & 6 \end{vmatrix}.$
6. $\begin{vmatrix} -\frac{1}{2} & 9 \\ \frac{1}{3} & 4 \end{vmatrix}.$
7. $\begin{vmatrix} \frac{1}{2} & -\frac{1}{8} \\ 2 & \frac{1}{3} \end{vmatrix}.$
8. $\begin{vmatrix} 2 & a \\ 3 & b \end{vmatrix}.$
9. $\begin{vmatrix} a & b \\ x & y \end{vmatrix}.$

98. The values of x and y in Section 95 may be expressed in Determinant Notation.

Thus,
$$x=\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}},\qquad y=\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

EXAMPLE

Solve:
$$\begin{cases} 4x+7y=-27, & (1) \\ x-2y=12. & (2) \end{cases}$$

$$x=\frac{\begin{vmatrix} -27 & 7 \\ 12 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 1 & -2 \end{vmatrix}}=\frac{-27\cdot(-2)-7\cdot 12}{4\cdot(-2)-7}=\frac{-30}{-15}=2. \tag{3}$$

$$y=\frac{\begin{vmatrix} 4 & -27 \\ 1 & 12 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 1 & -2 \end{vmatrix}}=\frac{4\cdot 12-1(-27)}{-15}=\frac{75}{-15}=-5. \tag{4}$$

TEST. $4\cdot 2+7(-5)=-27$; $2-2(-5)=12$.

99. This method usually gives the values by inspection, for the products and differences can be read direct from the equations themselves.

For example:

$4x+2y=1$, $3x-5y=4$. Denominators $=4(-5)-2\cdot 3$ $=-26$.	$4x+2y=1$, $3x-5y=4$. Numerator of y $=4\cdot 4-1\cdot 3=13$. $\therefore y=\frac{13}{-26}=-\frac{1}{2}$.	$4x+2y=1$, $3x-5y=4$. Numerator of x $=1(-5)-2\cdot 4=-13$. $\therefore x=\frac{-13}{-26}=\frac{1}{2}$.

WRITTEN EXERCISES

Solve, using determinant forms:

1. $x+y=5$, $x-y=3$.
2. $x+y=5$, $x-y=1$.
3. $2x+y=3$, $x+y=2$.
4. $2x+3y=7$, $x-y=1$.
5. $2x+2y=8$, $2x-y=2$.
6. $3x-y=-5$, $2x-y=-3$.

7. $4x - 3y = 7,$
$3x - 4y = 7.$

8. $5x + y = 9,$
$3x + y = 5.$

9. $4x + 5y = 22,$
$3x + 2y = 13.$

10. $x - 5y = -22,$
$5x - y = 10.$

11. $4x - 3y = 3,$
$3x - 4y = -3.$

12. $4x + 2y = 1,$
$3x - 2y = \frac{5}{2}.$

13. $12x - 11y = 87,$
$4x + 2y = 46.$

14. $7x - 2y = 3,$
$7x - 4y = -1.$

15. $9x - 3y = -6,$
$8x - 2y = -6.$

16. $ax + y = 1,$
$bx + y = 2.$

17. $ax + by = c,$
$px + qy = d.$

18. $x - my = a,$
$x + py = b.$

19 $ax - by = e,$
$cx - dy = f.$

20. $ax - y = b,$
$cx + y = d.$

21. $2x + 7y = 11,$
$5x - 9y = 1.$

22. $3x + 7y = -1,$
$2x - 3y = 7.$

23. $4x - 7y = 5,$
$8x + 15y = 39.$

24. $2x - y = 6,$
$10x - 3y = 7.$

25. $7x + 9y = 20,$
$8x - 4y = -20.$

26. $2x - 6y = 6,$
$4x + 2y = 16.$

27. $2x + y = 0,$
$x + 2y = -3.$

28. $11x + 22y = 33,$
$4x + 18y = 22.$

29. $1.4x + 2.1y = 1,$
$2.8x + 3.3y = 2.$

30. $30x + 25y = 40,$
$13x + 16y = 22\frac{1}{2}.$

31. $9x - 12y = -51,$
$21x - 35y = -133.$

32. $3x + 5y = 37,$
$10x - 5y = 15.$

33. $5x = 3y,$
$2x + 8y = 4.$

34. $ax - by = c,$
$cx + ay = b.$

35. $4a^2x + 5ay = 3,$
$6ax + 7y = 2.$

36. $ax + by = c,$
$a^2x + b^2y = c^2.$

37. $x + by = 1,$
$\frac{x}{a} + y = 3.$

38. $\frac{2}{3}x + \frac{5}{6}y = \frac{1}{2},$
$\frac{3}{8}x + \frac{7}{16}y = \frac{1}{8}.$

Note. See Chapter X for problems relating to geometry.

SPECIAL SYSTEMS

100. The study of graphs of systems of equations helps to interpret special cases.

101. We have solved (Sec. 95) the equations

$$ax + by = e,$$
$$cx + dy = f,$$

and found that $x = \frac{de - bf}{ad - bc};\ y = \frac{af - ec}{ad - bc}.$

I. Let us give to the letters a, b, c, d, such values that $ad - bc = 0$; for example, $a = 2$, $b = 1$, $c = 4$, $d = 2$. And let us give to e and f such values that $de - bf$ is not 0; for example, $e = 5$, $f = 4$.

Then the above results become:

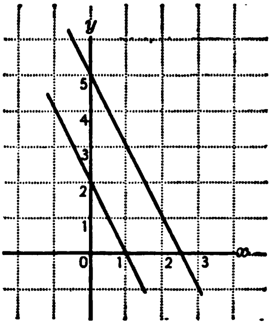
$$x = \frac{6}{0};\ y = -\frac{12}{0}.$$

The indicated division by zero means that the solution is impossible, Sec. 44. There is no pair of values that satisfies both equations. This appears readily also by substituting the values 2, 1, 4, 2, 5, 4, for a, b, c, d, e, f, in the given equations, which then become:

$$\left.\begin{aligned}2x + y &= 5,\\ 4x + 2y &= 4,\end{aligned}\right\}\quad\text{or}\quad\left\{\begin{aligned}2x + y &= 5,\\ 2x + y &= 2.\end{aligned}\right.$$

It is obvious that no set of values of x and y can make $2x + y$ equal to 5 and also equal to 2.

This condition can be illustrated graphically:



For drawing the graphs of

$$2x + y = 5 \text{ and } 2x + y = 2,$$

the two lines are parallel. That two parallel straight lines do not intersect is the geometric condition corresponding to the fact that a system of two incompatible equations has no solution.

The two equations are called **incompatible** or **contradictory.**

II. Retaining the values of a, b, c, d, above, let us give e and f such values that the numerators of the result both become zero; for example, $e=5, f=10$.

The result assumes the form:

$$x=\frac{0}{0};\ y=\frac{0}{0}.$$

Under these conditions we have seen, Section 44, that x and y may have any values. But this may also be seen by reference to the equations.

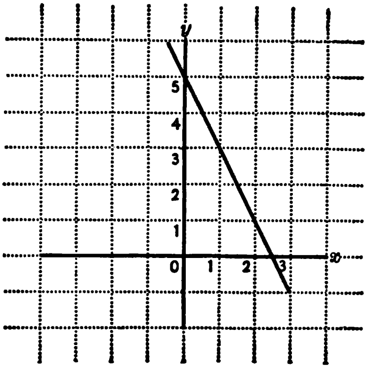
Substituting the values 2, 1, 4, 2, 5, 10, for a, b, c, d, e, f, in the given equations, they become:

$$2x+y=5.$$
$$4x+2y=10.$$

It appears that the second equation is twice the first, and hence equivalent to it. Any values of x and y that satisfy the first, will also satisfy the second.

We can choose arbitrarily any value for x and then determine a value of y to go with it by means of the first equation. For example, choosing $x=3$, then $2\cdot 3+y=5$, which gives $y=-1$. These values of x and y satisfy both equations. Similarly, any value can be chosen for y, and a value of x found such that the pair of values satisfies the given system.

The two equations are **dependent**. Every solution of one is a solution of the other.



If we undertake to make the graphs of the two equations as given, we find that they lead to the same straight line. The two graphs are coincident; every point of the straight line is a common point of the two graphs. Any abscissa x is the abscissa of a common point of the graphs; any ordinate y is the ordinate of a common point of the graphs.

NOTE. The study of expressions which may assume the exceptional forms mentioned above, especially those which may assume the form $\frac{0}{0}$, is very important, both from the point of view of later mathematics and the physical sciences.

102. Number of Solutions. We have thus seen that systems of two linear equations in two unknowns may be classified as follows:

1. Independent (the ordinary case, admitting one solution).
2. Contradictory (admitting no solution).
3. Dependent (admitting a boundless number of solutions).

WRITTEN EXERCISES

Construct the graphs of each of the following systems and classify them according to Section 102:

1. $3x + y = 2,$
   $x + y = 0.$
2. $2x - y = 1,$
   $4x - 2y = 2.$
3. $s - t = 6,$
   $s + t = 6.$
4. $x + 2z = 10,$
   $x + 3z = 11.$
5. $x = 25,$
   $y = 10.$
6. $10x + 5y = 25,$
   $2x + y = 5$
7. $7x + 14y = 7,$
   $x + 2y = 2.$
8. $12x - 3y = 8,$
   $3y - x = 4.$
9. $2x - 3y = -5,$
   $x + 2y = 8.$

EQUATIONS WITH THREE OR MORE UNKNOWNS

103. The definitions and methods for the solution of two equations with two unknowns may be applied equally well to a greater number of equations and unknowns.

To solve three linear equations with three unknowns, eliminate one unknown from any pair of the equations and the same unknown from any other pair; two equations are thus formed which involve only two unknowns and which may be solved by methods previously given.

Four or more equations with four or more unknowns may be solved similarly.

104. Determinants of the third and higher orders have been defined, and can be used to solve linear equations with three or more unknowns, but the method is too complicated to be of practical value here.

WRITTEN EXERCISES

Solve:

1. $4x-2y+z=3,\quad x+3y+2z=13,\quad -8x+12y+z=21.$

2. $5x+4y+2z=17,\quad 3x-2y+5z=2,\quad 2x-y+3z=2.$

3. $x-y-z=a,\quad 3y-x-z=2a,\quad 7z-y-x=4a.$

4. $3x-2z+5=0,\quad 2x+3y-21=0,\quad 4y+7z-69=0.$

5. $x+y=\frac{1}{a},\quad y+z=\frac{1}{b},\quad x+z=\frac{1}{c}.$

6. $$\frac{x}{a+b}+\frac{y}{b-c}+\frac{z}{a+c}=2c,$$
$$\frac{x}{a-b}-\frac{y}{b-c}-\frac{z}{a-c}=2a,$$
$$\frac{x}{a-b}+\frac{y}{c-b}-\frac{z}{a+c}=2a-2c.$$

7. $$x-y-z-2w=-12,$$
$$3x-y-2z+8w=40,$$
$$4x-4y+7z-5w=52,$$
$$3x-y+2z+w=44.$$

8. A and B can do a piece of work in 6 days; A and C can do it in 9 days, and A, B, and C can do 8 times the whole work in 45 days. In how many days can each do it alone?

9. A sum of money is divided into 3 parts such that the first part exceeds the second part by $100. The annual income from the second and third parts, if invested at 6% per annum, is $42. The sum of the first and second parts equals the sum of the second and third parts. Find the number of dollars in each part.

10. A number consists of 3 digits. If 99 is added to it the sum is a number having the same digits, but in reverse order. The sum of the hundreds' and tens' digits equals the units' digit, and the units' digit exceeds the tens' digit by 1.

11. The sum of the digits in a certain number of three figures is 13, the hundreds' digit exceeds the tens' digit by 1, and the units' digit exceeds the hundreds' digit by 2. Find the number.

12. Three casks together contain 79 gallons; the second contains 3 gallons more than $\frac{1}{2}$ as much as the first, and the third contains 7 gallons less than the second. How many gallons are there in each? (From a fourteenth century manuscript.)

Note. Those who have studied geometry may take up here some of the problems based upon geometric properties found in Chapter X.

QUADRATIC EQUATIONS

105. Quadratic Equations. Equations of the second degree are called **quadratic equations.**

A general form for quadratic equations in one unknown is

$$ax^2 + bx + c = 0,$$

in which a, b, c represent any known numbers, except that a may not be zero.

106. Solution of Quadratic Equations.

(1) The incomplete quadratic equation $x^2 = a$ is solved by extracting the square root of both members. The roots are: $x = \pm\sqrt{a}$.

(2) The incomplete quadratic equation $ax^2 + bx = 0$ is solved by factoring. The roots are $x = 0$ and $x = -\frac{b}{a}$.

(3) Complete quadratic equations are solved by completing the square.

The process consists of two main parts:

(*a*) *Making the left member a square while the right member does not contain the unknown.*

This is called completing the square.

It is based upon the relation $(x + a)^2 = x^2 + 2ax + a^2$, in which it appears that the last term, a^2, is the square of one half of the coefficient of x.

(*b*) *Extracting the square roots of both members and solving the resulting linear equations.*

Square roots which cannot be found exactly should be indicated.

EXAMPLE

Solve: $x^2 - 8x + 9 = 0.$ (*1*)

Transposing, $x^2 - 8x = -9.$ (*2*)

Completing the square, $x^2 - 8x + 16 = -9 + 16.$ (*3*)

Rearranging, $(x - 4)^2 = 7.$ (*4*)

Extracting the square root, $x - 4 = \pm\sqrt{7}.$ (*5*)

Solving (*5*) for x, $x = 4 \pm \sqrt{7}.$ (*6*)

(4) If any quadratic equation has zero for the right member, and if the polynomial constituting the left member can be factored, the quadratic is equivalent to two linear equations whose roots can readily be found. (See Chapter XIII, Part I.)

WRITTEN EXERCISES

Solve:

1. $3x^2 = 18.$

2. $x^2 - 5x = 0.$

3. $x^2 - 5x + 6 = 0.$

4. $x^2 + 4x - 3 = 0.$

5. $t^2 - 2t - 6 = 0.$

6. $8p^2 = 5p.$

7. $x^2 + 11x + 24 = 0.$

8. $5x^2 - 13x + 5 = 0.$

9. $15y^2 + 134y + 288 = 0.$

10. $\frac{x}{4} = \frac{9}{x-8} + \frac{2x}{5} + 1.35.$

11. $\frac{1}{y-3} + \frac{1}{1-y} - \frac{1}{y-2} = 0.$

12. $\frac{6-2w}{w-2} + \frac{5+w}{3+w} = \frac{w-5}{2-w}.$

13. $7(7-z)(z-6) + 3(5-z)(2-z) - 40 = 0.$

14. Find two numbers whose sum is 10, and the sum of whose squares is 68.

Suggestion. Let x represent one number, and $10 - x$ the other.

15. A room is 3 yd. longer than it is wide; at $1.75 per square yard, carpet for the room costs $49. Find the dimensions of the room.

16. A man bought for $300 a certain number of oriental rugs, each at the same price. If he had bought rugs each costing $40 more, he would have obtained 2 fewer rugs. How many rugs did he buy?

17. A dealer bought a number of similar tables for \$153. He sold all but 7 of them at an advance of \$1 each on their cost, thus receiving \$100. How many tables did he buy?

18. A man invested \$6000 at a certain rate of simple interest during 4 years. At the end of that time he reinvested the capital and the interest received during the 4 years at a rate of interest 1% lower than at first. His annual income from the second investment was \$372. What was the original rate of interest?

19. A rectangle whose area is 84 sq. in. is 5 in. longer than it is wide. Find its dimensions.

20. A certain number of men hire an automobile for \$156. Before they start, two others join them, sharing equally in the expense. The amount to be paid by each of the original renters is thus reduced by \$13. How many men were there at first?

21. A man rows down a stream a distance of 21 mi. and then rows back. The stream flows at 3 mi. per hour and the man makes the round trip in $13\frac{1}{8}$ hours. What is his rate of rowing in still water?

22. The product of a number and the same number increased by 40 is 11,700; what is the number?

23. If each side of a certain square is increased by 5 the area becomes 64; what is the length of a side?

24. Find two numbers whose sum is 16 and the difference of whose squares is 32.

25. A number multiplied by 5 less than itself is 750. Find the number.

26. The product of two consecutive even numbers is 728. Find the numbers.

INTERPRETATION OF RESULTS

107. Interpretation of Results. After the conditions of a problem have been expressed by equations, and the equations solved, the result must be examined to see whether it is admissible under the conditions of the problem.

EXAMPLES

1. Find three consecutive integers such that their sum shall be equal to 3 times the second.

Solution. 1. Let $x =$ the first.

2. Then, $x + 1 =$ the second,
3. and $x + 2 =$ the third.
4. $\therefore x + (x + 1) + (x + 2) = 3(x + 1)$.
5. $\therefore (3 - 3)(x + 1) = 0$, or $0(x + 1) = 0$.

Interpretation of the Result. The equation determines no particular value of x; it exists for every value of x. Consequently, every three consecutive integers must satisfy the given conditions.

2. Find three consecutive integers whose sum is 57, and the sum of the first and third is 40.

Solution. 1. Let $x =$ the first.

2. Then $x + 1 =$ the second,
3. and $x + 2 =$ the third.
4. Then, $x + (x + 1) + (x + 2) = 57$,
5. and $x + (x + 2) = 40$, by the given conditions.
6. From (4), $x = 18$.

Interpretation of the Result. $x = 18$ will not satisfy equation (5); therefore no three consecutive integers satisfy the problem.

3. The town B is d mi. from A; two trains leave A and B simultaneously, going in the same direction (that from A towards B), A at the rate of m mi. per hour and B q mi. per hour. How far from B will the trains be together?

Solving this problem by the usual method, we find as the result $\frac{qd}{m - q}$.

Interpretation of the Result. If d is not equal to 0, and if $m = q$, the result assumes the form $\frac{qd}{0}$. This means that the problem is impossible under these conditions. This is evident also from the meaning of m and q in the problem. If the two trains go in the same direction at the same rate, the one will always remain d miles behind the other.

If, however, $d = 0$, and $m = q$, the result assumes the form $\frac{0}{0}$, which equals any number whatever. This also agrees with the conditions of the problem. If d is zero, B and A are coincident, and the two trains are together at starting. If $m = q$, they both run at the same rate, and always remain together. They are therefore together at every distance from B.

WRITTEN EXERCISES

Solve and interpret the results:

1. Fifteen clerks receive together $150 per week; suppose that some receive $8 and others $12 per week. How many would there be receiving each salary?

2. A train starts from New York for Richmond via Philadelphia and Baltimore at the rate of 30 miles an hour, and two hours later another train starts from Philadelphia for Richmond at the rate of 20 miles an hour. How far beyond Baltimore will the first train overtake the second, given that the distance from New York to Philadelphia is 90 miles and from Philadelphia to Baltimore 96 miles?

3. If the freight on a certain class of goods is 2 cents per ton per mile, together with a fixed charge of 5 cents per ton for loading, how far can 2000 tons be sent for $80?

4. Find three consecutive integers whose sum equals the product of the first and the last.

5. The hot-water faucet of a bath tub will fill it in 14 minutes, the cold-water faucet in 10 minutes, and the waste pipe will empty it in 4 minutes. How long will it take to fill the tub when both faucets and the waste pipe are opened?

REVIEW

WRITTEN EXERCISES

Solve:

1. $\frac{x+3}{x-1}=\frac{x-1}{x+3}$.

2. $x^2-14x+33=0$.

3. $a+\frac{b}{x}=c$.

4. $2x+3(4x-1)=5(2x+7)$.

5. $14x-5(2x+4)=3(x+1)$.

6. $\frac{3x}{2}-\frac{4x}{3}=50$.

7. $\frac{x+150}{x+50}=\frac{6}{5}$.

8. $4x+19=5x-1$.

9. $4-\frac{3}{2x}=6$.

10. $3x-40=100-\frac{x}{2}$.

11. $\frac{x}{2}-2-\frac{x}{14}+\frac{2}{7}=24$.

12. $\frac{2x}{3}+4=\frac{4x}{5}-2.$

13. $2x-\frac{30-x}{2}=35.$

14. $\frac{5x}{16}+875=4000.$

15. $7x+4-15x=6x+2.$

16. $5x+4=9x+1.$

17. $\frac{x}{x+60}=\frac{7}{3x-5}.$

18. $\frac{2(2a-bx)}{3b^2}=\frac{a}{b}.$

19. $\frac{180}{2x+6}=\frac{x}{2}.$

20. $100x-x^2=-2400.$

21. $x(x+4)=45.$

22. $\frac{120}{x}-\frac{120}{x+3}=2.$

23. $x(7-x)=12.$

24. $x^2-8x=9.$

25. $\frac{x+7}{x}+\frac{x}{x+7}=\frac{5}{2}.$

26. $y^2+3y=70.$

27. $z^2-3z=70.$

28. $11m^2-10m=469.$

29. $\frac{90}{r}-\frac{27}{r+2}=\frac{90}{r+1}.$

30. $\frac{12}{5-x}+\frac{8}{4-x}=\frac{32}{x+2}.$

31. $2y-3y+120$
$=4y-6y+132.$

32. $7x^2-10x=120+4x^2-19x.$

33. $(5x-2)(6x+1)-(10x+3)(3x+10)=0.$

34. $(2x-3)(x+1)-(3x-7)(x-4)=36.$

35. $\frac{1}{x+2}+\frac{2}{x-2}=\frac{7}{2x-3}.$

36. $2x+5y=6,$
$4x+11y=3.$

37. $5x-3y=1,$
$13x-8y=9.$

38. $\frac{x}{y}=7,$
$2x-10y=3y+2.$

39. $\frac{7y-6}{3x+4}=\frac{1}{2},$
$\frac{4y-5}{2x+1}=\frac{1}{3}.$

40. $\frac{3x+2y+5}{2}+\frac{x-3y-13}{5}+\frac{y-3x+3}{6}=3,$
$\frac{2x-4y+6}{y-5x+11}=-2.$

41. $6z=43-5y,$
$3z=37-4x,$
$4y=55-5x.$

42. $3x+4y+2z=-4,$
$2x-5y-z=9,$
$-4x+2y+3z=-23.$

43. $\frac{y-z}{x+y}=\frac{1}{5}, \frac{x-z}{y+z}=\frac{2}{3}, \frac{x-y}{x+z}=\frac{1}{4}.$

44. $(z-2)(x+3)=(x-1)(z-1),$
$(z+8)(y-2)-(y+2)(z+2)=0,$
$y(3-2x)+(2y-3)(1+x)=0.$

45. $\frac{3}{2x+10}-\frac{3x^2+14}{7(4-x)}+\frac{1+3x}{5+x}-\frac{3x}{7}=0.$

46. $2(2x+3y)-\frac{5(x+3)}{8}-\frac{3y}{4}=9, x+y=1.$

47. A rectangle whose length is greater than its breadth by 1 yd. has an area of 6 sq. yd. Find its dimensions.

48. A square $(2x-3)$ ft. on a side has taken from it a square x ft. on a side. The remaining area is 24 sq. ft.; find the side of each square.

49. The product of two consecutive numbers is 380; find the numbers.

50. The product of two consecutive even numbers is 840. Represent the smaller by $2n$ and find both numbers.

51. In a certain election 36,785 votes were cast for the three candidates A, B, C. B received 812 votes more than twice as many as A; and C had a majority of one vote over A and B together. How many votes did each receive?

52. In a certain election there were two candidates, A and B. A received 10 votes more than half of all the votes cast. B received 4 votes more than one third of the number received by A. How many votes did each receive?

53. A group of friends went to dine at a certain restaurant. The head waiter found that if he were to place five persons at each table available, four would have no seats, but by placing six at each table, only three persons remained for the last table. How many guests were there, and how many tables?

54. A flower bed of uniform width is to be laid out around a rectangular house 20 ft. wide and 36 ft. long. What must be the width of the bed in order that its area may be one third of that of the ground on which the house stands?

55. Wood's metal, which melts in boiling water, is made up of one half (by weight) of bismuth, a certain amount of lead, half that much zinc, and half as much cadmium as zinc. How much of each is there in 100 lb. of Wood's metal?

56. If in the preceding exercise $\frac{3}{4}$ as much cadmium as zinc is used, a different metal is formed. How many pounds of each constituent metal in 100 lb. of this metal?

57. A cask contains 10 gal. of alcohol. A certain number of quarts are drawn out; the cask is then filled up with water and the contents thoroughly mixed. Later, twice as many quarts are drawn out as the previous time and the cask filled up with water. There now remain only 4.8 gal. of alcohol in the mixture. How many gallons were drawn out at first?

SUGGESTION. Let $x =$ the number of gallons first withdrawn. Then $10 - x =$ the number of gallons left.

When the cask is filled again with water any part of the mixture is $\frac{10-x}{10}$ alcohol. Then, $\frac{10-x}{10}$ of the $2x$ gallons of mixture withdrawn the second time is alcohol. Hence, $10 - x - \left(\frac{10-x}{10}\right)2x$ is the number of gallons of alcohol left in the cask.

58. How much water must be added to 30 oz. of a 6 % solution of borax to make a 4 % solution?

59. How much acid must be added to 10 quarts of a 2 % solution to make a 5 % solution?

60. Alcides was asked, "How many are there of your numerous herd?" He replied: "If I had 6 less than twice as many more, the number would be 306. Find the number."

61. A courier went from Paris to Grenoble, 120 leagues, in 4 days, each day's journey being 2 leagues shorter than that of the preceding day. How many leagues did he travel each day? (Ozanam's *Algebra*, 1702.)

62. Two messengers, A and B, set out towards each other from two places 59 mi. apart, B starting 1 hr. after A. A goes 7 mi. in 2 hr., and B 8 mi. in 3 hr. How far will A have gone when he meets B? (Newton's *Arithmetica Universalis*, 1707.)

63. A merchant bought a certain number of platters for \$366. Three were broken during shipment. He sold $\frac{1}{8}$ of the remainder for \$75 at a profit of 25%. Find the number of platters bought and the price per platter.

64. A certain hall contains both gas jets and electric lights. When 60 gas jets and 80 electric lights are used, the cost for an evening is \$4. If 90 gas jets and 60 electric lights are used, the cost is \$4.05. Find the cost per gas jet and electric light.

65. A tailor paid \$12 for 4 yd. of cloth and 8 yd. of lining. At another time he paid \$21 for 6 yd. of the cloth and 16 yd. of the lining. Find the price of each per yard.

66. Two wheelmen are 328 ft. apart and ride toward each other. If A starts 3 seconds before B, they meet in 14 seconds after A starts; or if B starts 2 seconds before A, they meet in 14 seconds after B starts. Find the rate of each.

67. A man had a portion of his capital invested in stocks paying 6% dividends, the remainder in mortgages paying 5%. His annual income was \$700. The next year the dividend on the stock was reduced to 5%, but by reinvestment he replaced his old mortgages by new ones paying $5\frac{1}{2}$%. His income for this year was \$690. How much had he invested in stocks? Also in mortgages?

68. A company at a tavern, when they came to pay, found that if the same bill were divided among three persons more, the amount would be one shilling less per person; and, if it were divided among two persons fewer, it would be one shilling more per person. Find the number of persons in the original company, and the amount of the bill. (Saunderson's *Algebra*, 1740.)

69. One person says to another, "If you give me three of your coins, I shall have as many as you." The second person replies, "If you give me three of yours, I shall have twice as many as you have." Find the numbers that each has. (Ozanam's *Algebra*, 1702.)

70. A coach set out from Cambridge to London with four more passengers outside than within. Seven outside passengers could travel at 2 shillings less expense than 4 inside passengers. The fares of all the passengers amounted to 180 shillings. At the end of half the journey the coach took up 1 more inside and 3 more outside passengers; these paid $\frac{2}{15}$ as much as the others. Required the number of passengers and the fare of each. (Bland's *Algebraical Problems*, 1816.)

71. Seven years ago a man was 4 times as old as his son; 7 years hence he will be only double his age. Find the age of each. (Simpson's *Algebra*, 1767.)

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. State the general form of a linear equation with one unknown, and the formula for its root. Secs. 86–89.

2. When is a system of equations simultaneous? Sec. 93.

3. State the general form of a system of two linear equations with two unknowns. Sec. 95.

4. State two methods of solving simultaneous equations. Secs. 94, 95.

5. What is the general form of the values of the unknowns in simultaneous equations with two unknowns? Sec. 95.

6. What use may be made of determinants in solving simultaneous equations? Secs. 97, 98.

7. What is the general form of quadratic equations with one unknown? Sec. 105.

8. Name two kinds of quadratic equations, and the method of solving each. Sec. 106.

CHAPTER III

EXPONENTS AND ROOTS

LAWS OF EXPONENTS

108. PREPARATORY

1. What is the meaning of a^2? Of a^4? Of a^n?

2. What is the meaning of $\sqrt{a^2}$? Of $\sqrt[3]{a^3}$? Of $\sqrt[n]{a^n}$?

3. $a^2 \cdot a^3 = ?$ $a^3 \cdot a^2 = ?$ $a^5 \cdot a^3 = ?$ $a^8 \cdot a^5 = ?$

4. $a^3 \div a^2 = ?$ $a^4 \div a^2 = ?$ $a^5 \div a^2 = ?$ $b^{10} \div b^4 = ?$

5. $(a^2)^2 = ?$ $(a^2)^3 = ?$ $(c^5)^2 = ?$ $(x^{10})^3 = ?$

109. We shall soon define negative and fractional exponents, but until this is done literal exponents are to be understood to represent positive integers.

110. Law of Exponents in Multiplication.

I. $$a^m \cdot a^r = a^{m+r}.$$

For $a^m = a \cdot a \cdot a \cdots$ to m factors,

and $a^r = a \cdot a \cdot a \cdots$ to r factors.

$$\begin{aligned} \therefore a^m \cdot a^r &= (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})(a \cdot a \cdot a \cdots \text{ to } r \text{ factors}) \\ &= a \cdot a \cdot a \cdot a \cdots \text{ to } m + r \text{ factors} \\ &= a^{m+r}, \text{ by the definition of exponent.} \end{aligned}$$

Similarly, $$a^m \cdot a^r \cdot a^p \cdots = a^{m+r+p} \cdots.$$

ORAL EXERCISES

Multiply:

1. $a^2 \cdot a^4$.
2. $a^3 \cdot a^3$.
3. $a^5 \cdot a^7$.
4. $m^1 \cdot m^5$.
5. $x \cdot x$.
6. $2^3 \cdot 2^4$.
7. $(-1)^3 \cdot (-1)^5$.
8. $6^2 \cdot 6^2$.
9. $5 \cdot 5 \cdot 5^2$.
10. $2^3 \cdot 2^3 \cdot 2^2$.
11. $7 \cdot 7^2 \cdot 7^3$.
12. $3 \cdot 3^5 \cdot 3^2$.
13. $(-1)^2 \cdot (-1)^3 \cdot (-1)^5$.
14. $(-a)^2 \cdot (-a)^4 \cdot (-a)$.

111. Law of Exponents in Division.

II. $$\frac{a^m}{a^r} = a^{m-r}, \text{ if } m > r.$$

For $a^m = a \cdot a \cdot a \cdots m$ factors,

and $a^r = a \cdot a \cdot a \cdots r$ factors.

$$\therefore \frac{a^m}{a^r} = \frac{a \cdot a \cdots m \text{ factors}}{a \cdot a \cdots r \text{ factors}}$$

$= a \cdot a \cdot a \cdots m - r$ factors, canceling the r factors from both terms

$= a^{m-r}$, by definition of exponent.

ORAL EXERCISES

Divide:

1. $\frac{a^4}{a^2}$.
2. $\frac{a^5}{a^3}$.
3. $\frac{a^7}{a^5}$.
4. $\frac{2^4}{2^2}$.
5. $\frac{a^8}{a^5}$.
6. $\frac{(-a)^5}{(-a)}$.
7. $\frac{(-1)^5}{(-1)^3}$.
8. $\frac{(ab)^2}{(ab)}$.
9. $\frac{6^3}{6^2}$.
10. $\frac{5^4}{5^3}$.
11. $\frac{4\pi r^2}{4r}$.
12. $\frac{x^5}{x}$.
13. $\frac{x^2y}{x}$.
14. $\frac{mv^2}{v}$.
15. $\frac{\frac{1}{2}gt^2}{t}$.
16. $\frac{\pi r^3}{r}$.

112. Laws of Exponents for Powers.

III. $$(a^m)^r = a^{mr}.$$

For $(a^m)^r = a^m \cdot a^m \cdot a^m \cdots$ to r factors

$= (a \cdot a \cdots$ to m factors$)(a \cdot a \cdots$ to m factors$)$ to r such parentheses

$= a \cdot a \cdot a \cdots$ to mr factors

$= a^{mr}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(4^3)^2$.
2. $(3^2)^5$.
3. $(2^5)^4$.
4. $(a^3)^2$.
5. $(a^2)^3$.
6. $(a^5)^2$.
7. $(x^2)^5$.
8. $(x^3)^3$.
9. $(y^4)^4$.
10. $[(b)^4]^5$.
11. $[(-a)^5]^2$.
12. $[(-8)^2]^3$.

IV. $$(ab)^n = a^n b^n.$$

For $(ab)^n = (ab) \cdot (ab) \cdots$ to n factors

$= (a \cdot a \cdots \text{ to } n \text{ factors})(b \cdot b \cdots \text{ to } n \text{ factors})$

$= a^n b^n$, by definition of exponent.

Similarly, $(abc \cdots)^n = a^n b^n c^n \cdots$.

ORAL EXERCISES

Apply this law to:

1. $(8 \cdot 3)^2$.
2. $(4 \cdot 5)^2$.
3. $(2 \cdot 5)^3$.
4. $(ab)^5$.
5. $(cd)^3$.
6. $(abc)^4$.
7. $(mn)^p$.
8. $(xy)^a$.
9. $(ab)^{2n}$.
10. $(x^2y)^3$.
11. $(x^a y^3)^5$.
12. $(ax^3)^b$.

V. $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

For $\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdots$ to n factors

$= \frac{a \cdot a \cdots \text{ to } n \text{ factors}}{b \cdot b \cdots \text{ to } n \text{ factors}}$

$= \frac{a^n}{b^n}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(\frac{2}{3})^2$.
2. $(\frac{1}{5})^3$.
3. $(\frac{2}{5})^5$.
4. $(\frac{3}{5})^4$.
5. $(-\frac{6}{7})^2$.
6. $(-\frac{3}{4})^3$.
7. $\left(\frac{a}{b}\right)^2$.
8. $\left(\frac{a}{b}\right)^5$.
9. $\left(\frac{c}{d}\right)^3$.
10. $\left(\frac{x}{y}\right)^n$.
11. $\left(\frac{m}{n}\right)^a$.
12. $\left(\frac{p}{q}\right)^m$.
13. $\left(\frac{2a}{b}\right)^5$.
14. $\left(\frac{c}{5d}\right)^7$.
15. $\left(\frac{-xy}{z}\right)^5$.

113. Collected Laws of Exponents.

I. $a^m \cdot a^r = a^{m+r}$. Sec. 110.
II. $a^m \div a^r = a^{m-r}$. $(m > r.)$ Sec. 111.
III. $(a^m)^r = a^{mr}$. Sec. 112.
IV. $(ab)^n = a^n b^n$. Sec. 112.
V. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. Sec. 112.

FRACTIONAL EXPONENTS

114. Hitherto we have spoken only of positive integers as exponents, the exponent meaning the number of times the base is used as a factor. This meaning does not apply to fractional and negative exponents, because it does not mean anything to speak of using a as a factor $\frac{2}{3}$ of a time, or -6 times. But it is possible to find meanings for fractional and negative exponents such that they will conform to the laws of integral exponents.

115. Preparatory.

Find the meaning of $a^{\frac{1}{2}}$.

Assuming that Law I applies, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$,

or $$(a^{\frac{1}{2}})^2 = a.$$

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a,

or, $$a^{\frac{1}{2}} = \sqrt{a}.$$

Thus, the fractional exponent $\frac{1}{2}$ means *square root.*

Similarly, $$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a,$$

or, $$(a^{\frac{1}{3}})^3 = a.$$

That is, $a^{\frac{1}{3}}$ is one of the three equal factors of a,

or, $$a^{\frac{1}{3}} = \sqrt[3]{a}.$$

Thus, the fractional exponent $\frac{1}{3}$ means *cube root.*

WRITTEN EXERCISES

Find similarly the meaning of:

1. $a^{\frac{1}{4}}$.
2. $a^{\frac{1}{5}}$.
3. $x^{\frac{1}{6}}$.
4. $b^{\frac{1}{10}}$.
5. $n^{\frac{1}{8}}$.
6. $c^{\frac{1}{7}}$.
7. $m^{\frac{1}{5}}$.
8. $a^{\frac{1}{20}}$.

116. The meaning of $a^{\frac{1}{n}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots \text{to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots \text{to } n \text{ terms}}$$
$$= a^{n \cdot \frac{1}{n}}$$
$$= a^1 = a.$$

That is, $a^{\frac{1}{n}}$ is one of n equal factors of a, or $a^{\frac{1}{n}} = \sqrt[n]{a}$.

117. Preparatory.

Find the meaning of $a^{\frac{2}{3}}$.

Assuming that Law I applies, $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$, or $(a^{\frac{2}{3}})^3 = a^2$.

That is, $a^{\frac{2}{3}}$ is one of the three equal factors of a^2, or $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

118. The meaning of $a^{\frac{p}{q}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots \text{to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}\cdots \text{to } q \text{ terms}}$$
$$= a^{q \cdot \frac{p}{q}} = a^p.$$

That is, $a^{\frac{p}{q}}$ is one of the q equal factors of a^p,

or, $$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Similarly, $$a^{\frac{p}{q}} = a^{\frac{1}{q} \cdot p} = (\sqrt[q]{a})^p.$$

In words:

a *with the exponent* $\frac{p}{q}$ *denotes the* q*th root of the* p*th power of* a, *or the* p*th power of the* q*th root of* a.

This definition applies when p and q are positive integers. The meaning of negative fractional exponents is found in Section 127, p. 76.

ORAL EXERCISES

State the meaning of:

1. $5^{\frac{1}{3}}$. 3. $6^{\frac{1}{2}}$. 5. $4^{\frac{5}{7}}$. 7. $8^{\frac{2}{3}}$.

2. $a^{\frac{1}{2}}$. 4. $a^{\frac{2}{3}}$. 6. $b^{\frac{5}{6}}$. 8. $c^{\frac{4}{3}}$.

Find the value of:

9. $8^{\frac{4}{3}}$. 10. $16^{\frac{3}{4}}$. 11. $25^{\frac{3}{2}}$. 12. $32^{\frac{3}{5}}$.

WRITTEN EXERCISES

Express with fractional exponents:

1. $\sqrt[4]{a^3}$.
2. $\sqrt[5]{a^2}$.
3. $\sqrt[3]{mn}$.
4. $\sqrt{a}\sqrt[3]{b}$.
5. $\sqrt[3]{ab}$.
6. $\sqrt[3]{ab^2xy}$.
7. $\sqrt[3]{\dfrac{x^4y^2}{4}}$.
8. $\sqrt{\dfrac{9\,a^2b}{x}}$.
9. $\sqrt[8]{16\,x^4y^8}$.
10. $\sqrt{m}\sqrt[3]{n^2}\sqrt[5]{p}$.
11. $\sqrt[5]{\dfrac{32\,a^5b^8}{c^{15}}}$.
12. $\sqrt[3]{\dfrac{16\,a^2}{9\,b^2}}$.
13. $\sqrt[3]{\dfrac{-8\,x^3y^6}{5}}$.
14. $\sqrt[6]{\dfrac{64\,x^{12}}{y}}$.
15. $\sqrt{a+b}$.
16. $\sqrt{a^3+b^3}$.
17. $\sqrt[3]{b^2}\cdot\sqrt{a}$.
18. $\sqrt[5]{-a}\cdot\sqrt[5]{-b}$.
19. $\sqrt[3]{(a-b)^2}$.
20. $\sqrt[n]{a}$.
21. $\sqrt[n]{a}\cdot\sqrt[m]{b}$.
22. $\sqrt[n]{a^m}$.
23. $\sqrt[m]{a^n}$.
24. $\sqrt[pn]{b^{2n}}$.
25. $\sqrt[2m]{a^{mn}}$.
26. $\sqrt[2mn]{a^{2mn}}$.
27. $\sqrt[mn]{a^nb^m}$.
28. $\sqrt[n]{a}\cdot\sqrt[p]{q}$.
29. $\sqrt[2n]{abc}$.
30. $\sqrt[p]{a^q}\cdot\sqrt[q]{a^p}$.

119. The definition of positive fractional exponents has been found as a consequence of the assumption that Law I applies to them. It can be shown that the other laws of Section 113, p. 68, also apply to this class of exponents, as thus defined, and we shall so apply them, although the proof is omitted here.

120. According to Law I (Sec. 113), *when the bases are the same, the exponent of the product is found by adding the exponents.*

For example: $a^{\frac{1}{2}} \cdot a^{\frac{3}{4}} = a^{\frac{1}{2}+\frac{3}{4}} = a^{\frac{5}{4}}.$

A general formula for this statement is,

$$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = a^{\frac{mq}{nq}+\frac{np}{nq}} = a^{\frac{mq+np}{nq}}.$$

The number a, or the base, must be the same in all factors. When it is not, as in $a^{\frac{1}{2}} \cdot b^{\frac{3}{4}}$, the product cannot be found by adding the exponents.

WRITTEN EXERCISES

Find the products:

1. $a^{\frac{1}{2}} \cdot a^{\frac{2}{3}}$.
2. $4^2 \cdot 4^{\frac{2}{3}}$.
3. $7^{\frac{1}{2}} \cdot 7^{\frac{2}{3}}$.
4. $a^{\frac{2}{3}} \cdot a^{\frac{3}{5}}$.
5. $b^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
6. $a^{\frac{2}{3}} \cdot a^{\frac{3}{4}}$.
7. $m^{\frac{3}{4}} \cdot m^{\frac{2}{3}}$.
8. $a^{\frac{5}{6}} \cdot a^{\frac{1}{2}}$.
9. $r^{\frac{1}{n}} \cdot r^{\frac{1}{n}}$.
10. $a^{\frac{1}{n}} \cdot a^{\frac{1}{m}}$.
11. $x^{\frac{1}{a}} \cdot x^{\frac{1}{b}}$.
12. $x^{\frac{m}{n}} \cdot x^{\frac{p}{n}}$.
13. $p^{\frac{2}{3}} \cdot p^{\frac{1}{2}} \cdot p$.
14. $a^{\frac{4}{5}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{1}{2}}$.

121. According to Law II (Sec. 113), *when the bases are the same, the exponent of the quotient is found by taking the difference between the exponents.*

For example: $a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}.$

$a \div a^{\frac{3}{4}} = a^{1-\frac{3}{4}} = a^{\frac{1}{4}}.$

A general formula for this statement is,

$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}} = a^{\frac{mq-np}{nq}}.$$

$\frac{m}{n}$ is here supposed to be greater than $\frac{p}{q}$, but this restriction will be removed later.

WRITTEN EXERCISES

Find the quotients:

1. $a^{\frac{3}{4}} \div a^{\frac{1}{2}}$.
2. $a \div a^{\frac{7}{12}}$.
3. $a \div a^{\frac{1}{n}}$.
4. $a \div a^{\frac{p}{q}}$.
5. $a^{\frac{4}{5}} \div a^{\frac{3}{4}}$.
6. $x^{\frac{7}{8}} \div x^{\frac{3}{8}}$.
7. $a^{\frac{3}{4}} \div a^{\frac{2}{3}}$.
8. $x^{\frac{2}{5}} \div x^{\frac{1}{4}}$.
9. $ab \div (ab)^{\frac{1}{4}}$.
10. $\left(\frac{1}{a}\right)^{\frac{5}{6}} \div \left(\frac{1}{a}\right)^{\frac{1}{2}}$.
11. $5^{\frac{2}{3}} \div 5^{\frac{1}{2}}$.
12. $x^{\frac{m}{n}} \div x^{\frac{p}{n}}$.
13. $m^{\frac{1}{a}} \div m^{\frac{1}{b}}$.
14. $6^{\frac{2}{3}} \div 6^{\frac{1}{6}}$.
15. $m^{\frac{3}{8}} \div m^{\frac{1}{8}}$.
16. $p^{\frac{4}{5}} \div p^{\frac{3}{4}}$.

122. According to Law III (Sec. 113), *when an exponent is applied to a base having an exponent, the product of the exponents is the exponent of the result.*

For example:

$$(a^{\frac{1}{2}})^2 = a^{2 \cdot \frac{1}{2}} = a^1 = a.$$

$$(a^2)^{\frac{1}{4}} = a^{2 \cdot \frac{1}{4}} = a^{\frac{2}{4}} = a^{\frac{1}{2}}.$$

$$(a^{\frac{1}{3}})^{\frac{2}{5}} = a^{\frac{1}{3} \cdot \frac{2}{5}} = a^{\frac{2}{15}}.$$

A general formula for this statement is,

$$(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{m}{n} \cdot \frac{p}{q}} = a^{\frac{mp}{nq}}.$$

ORAL EXERCISES

Simplify by applying Law III:

1. $(2^{\frac{1}{2}})^{\frac{1}{3}}$.
2. $(3^{\frac{1}{2}})^2$.
3. $(3^{\frac{2}{3}})^3$.
4. $(5^{\frac{3}{4}})^2$.
5. $(b^{\frac{1}{3}})^{\frac{1}{2}}$.
6. $(a^{\frac{1}{2}})^{\frac{1}{3}}$.
7. $(a^{\frac{2}{3}})^{\frac{5}{6}}$.
8. $(a^{\frac{3}{4}})^{\frac{2}{3}}$.
9. $(x^{\frac{5}{6}})^{\frac{7}{8}}$.
10. $(y^9)^{\frac{1}{3}}$.
11. $(5^{\frac{1}{2}})^{\frac{2}{3}}$.
12. $(3^{\frac{2}{3}})^{\frac{1}{3}}$.
13. $(a^{\frac{1}{5}})^{\frac{1}{2}}$.
14. $(a^{\frac{3}{4}})^4$.
15. $(10^{\frac{1}{6}})^{\frac{1}{3}}$.
16. $[(a+b)^{\frac{1}{2}}]^{\frac{1}{2}}$.
17. $[(a-b)^{\frac{1}{3}}]^{\frac{3}{4}}$.
18. $[(a^2-b^2)^{\frac{1}{4}}]^{\frac{1}{5}}$.
19. $[(x^n-y^n)^{\frac{1}{p}}]^{\frac{p}{q}}$.

123. According to Law IV (Sec. 113), *an exponent affecting a product is applied to each factor,* and according to Law V (Sec. 113), *an exponent affecting a fraction is applied to both numerator and denominator.*

For example:

$$(8x^6y^2z)^{\frac{1}{3}} = 8^{\frac{1}{3}}x^{\frac{6}{3}}y^{\frac{2}{3}}z^{\frac{1}{3}} = 2x^2y^{\frac{2}{3}}z^{\frac{1}{3}}.$$

$$(a^m b^{\frac{m}{n}} c)^{\frac{p}{q}} = a^{\frac{mp}{q}} \cdot b^{\frac{mp}{nq}} \cdot c^{\frac{p}{q}}.$$

$$\left(\frac{x^2}{y^8}\right)^{\frac{1}{4}} = \frac{(x^2)^{\frac{1}{4}}}{(y^8)^{\frac{1}{4}}} = \frac{x^{\frac{2}{4}}}{y^{\frac{8}{4}}} = \frac{x^{\frac{1}{2}}}{y^2}.$$

A general formula for an exponent affecting a fraction is,

$$\left(\frac{a^{\frac{m}{n}}}{b^{\frac{p}{q}}}\right)^{\frac{r}{s}} = \frac{a^{\frac{mr}{ns}}}{b^{\frac{pr}{qs}}}.$$

WRITTEN EXERCISES

Simplify by applying Law IV:

1. $(a^3b^2)^{\frac{1}{2}}$.
2. $(a^2b^{\frac{1}{3}})^3$.
3. $(a^mb^n)^{\frac{1}{p}}$.
4. $(x^{\frac{1}{2}}y^{\frac{1}{4}})^8$.
5. $(16\,x^4y^8)^{\frac{1}{4}}$.
6. $(a^6bn^{9m})^{\frac{1}{3}}$.
7. $(27\,a^{12}b^9c^6)^{\frac{1}{3}}$.
8. $(m^{\frac{1}{2}}n^{\frac{1}{4}}p^{\frac{1}{5}})^{20}$.
9. $(86\,x^6y)^{\frac{1}{3}}$.
10. $\left(\frac{32\,a^5b^{10}}{c^{15}}\right)^{\frac{1}{5}}$.
11. $\left(\frac{64\,x^{12}}{y}\right)^{\frac{1}{6}}$.

124. When the bases are different and the fractional exponents are different, the exponents must be made to have a common denominator, before any simplification is possible.

For example: $a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = (a^3b^2)^{\frac{1}{6}}$.

A general formula for this statement is,

$$a^{\frac{m}{n}}b^{\frac{p}{q}} = a^{\frac{mq}{nq}} \cdot b^{\frac{np}{nq}} = (a^{mq} \cdot b^{np})^{\frac{1}{nq}}.$$

This is *simplifying by reducing exponents to the same order.*

WRITTEN EXERCISES

Simplify by reducing the exponents to the same order:

1. $a^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
2. $a^{\frac{2}{3}} \cdot b^{\frac{3}{4}}$.
3. $v^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
4. $a^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
5. $a^{\frac{1}{3}} \cdot b^{\frac{3}{4}}$.
6. $b^{\frac{1}{2}} \cdot 6^{\frac{1}{6}}$.
7. $5^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
8. $a^{\frac{1}{2}} \div b^{\frac{1}{3}}$.
9. $m^{\frac{1}{3}} \cdot n^{\frac{1}{2}} \cdot p^{\frac{1}{4}}$.
10. $m^{\frac{2}{3}} \cdot n^{\frac{3}{5}} \div p^{\frac{1}{2}}$.
11. $p^{\frac{1}{n}} \cdot q^{\frac{1}{m}} \cdot r^{\frac{1}{3}}$.
12. $x^{\frac{p}{q}} \cdot y^{\frac{m}{n}} \cdot z^{\frac{1}{n}}$.

125. It is usually preferable to indicate roots by fractional exponents instead of by radical signs, since operations are thus more easily seen.

COMPARISON

	By Radicals	By Exponents
1.	$\sqrt{a}\sqrt[3]{a} = \sqrt[6]{a^3}\sqrt[6]{a^2} = \sqrt[6]{a^3a^2} = \sqrt[6]{a^5}$.	$a^{\frac{1}{2}}a^{\frac{1}{3}} = a^{\frac{3}{6}}a^{\frac{2}{6}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}$.
2.	$\sqrt{b} \div \sqrt[3]{b} = \sqrt[6]{b^3} \div \sqrt[6]{b^2} = \sqrt[6]{b^3 \div b^2} = \sqrt[6]{b}$.	$b^{\frac{1}{2}} \div b^{\frac{1}{3}} = b^{\frac{3}{6}} \div b^{\frac{2}{6}} = b^{\frac{3}{6}-\frac{2}{6}} = b^{\frac{1}{6}}$.

WRITTEN EXERCISES

Simplify by use of fractional exponents as in the examples above:

1. $2^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$.
2. $\sqrt{\frac{49}{121}}$.
3. $\sqrt{8} \cdot 3\sqrt{2}$.
4. $2\sqrt{3} \cdot 3 \cdot \sqrt{10}$.
5. $2\sqrt[3]{3} \cdot 3\sqrt[3]{2}$.
6. $5^{\frac{1}{2}} \cdot 3 \cdot 5^{\frac{1}{3}}$.
7. $\sqrt{7} \cdot 11^{\frac{1}{2}}$.
8. $-8\sqrt{2} \cdot 12\sqrt{3}$.
9. $\sqrt{81a^{10}}$.
10. $\sqrt{5} \cdot \sqrt{75}$.
11. $(16\,x^4p)^{\frac{1}{4}}$.
12. $\sqrt[5]{32\,a^3b^{10}}$.
13. $\sqrt{36\,m^4n^2}$.
14. $\sqrt{64\,m^3n^6}$.
15. $(49\,a^4b^6)^{\frac{1}{2}}$.
16. $\sqrt[3]{2\,a^2b^2}$.
17. $3^{\frac{1}{2}} \cdot 2 \cdot 7^{\frac{1}{2}}$.
18. $3 \cdot 5^{\frac{1}{2}} \cdot 2\sqrt{3}$.
19. $\left(\frac{49\,a^4}{64\,b^6}\right)^{\frac{1}{2}}$.
20. $\sqrt[3]{a^3\sqrt{25\,b}}$.
21. $\sqrt{x^2y\sqrt[4]{25\,yz}}$.
22. $\sqrt[6]{a\sqrt[5]{ab^3}}$.
23. $\sqrt{\sqrt[3]{4\,x^2}}$.

24. $-\sqrt{12}\cdot 2\sqrt{3}$.

25. $a\sqrt{b}\cdot b\sqrt{a}$.

26. $(\sqrt{2}-\sqrt{3})2\sqrt{3}$.

27. $3^{\frac{1}{2}}(6^{\frac{1}{2}}-2\cdot 5^{\frac{1}{2}})$.

28. $(a\sqrt[3]{x})^2$.

29. $\sqrt{a^6b^8}$.

30. $\sqrt[3]{8a^5b^6}$.

31. $(27xy^2)^{\frac{1}{3}}$.

32. $\sqrt{32}\cdot\sqrt{2}$.

33. $(a^2b\sqrt[4]{xy^2})^3$.

34. $(m^3\sqrt[5]{p^4})^4$.

35. $\sqrt{6ab}\cdot\sqrt{2a}$.

36. $[2a(4a^2)^{\frac{1}{3}}]^2$.

37. $11^{\frac{1}{2}}\cdot 11^{\frac{1}{3}}\cdot 11^{\frac{1}{7}}$.

38. $\sqrt[3]{m^2\sqrt[3]{m^2}}$.

39. $\sqrt{\sqrt[4]{8x^3}}$.

40. $\sqrt{\sqrt[5]{81a^4}}$.

41. $\sqrt[3]{x^2\sqrt[3]{9a^2}}$.

42. $\sqrt[3]{x^3y^6\sqrt[4]{27a^3}}$.

43. $a(a^2b)^{\frac{1}{3}}\cdot b(ab^2)^{\frac{1}{3}}$.

44. $3\sqrt{9a}\cdot 3^{\frac{1}{2}}$.

45. $5\sqrt[4]{a^3x^3}\cdot 2\sqrt[3]{ax}$.

46. $3\sqrt{8}\cdot 2\sqrt[3]{6}\cdot 3\sqrt[4]{54}$.

MEANING OF ZERO AND NEGATIVE EXPONENTS

126. The meaning of a^0 may be found as follows:

Assuming that Law I holds for a^0, $a^5\cdot a^0 = a^{5+0}$

$$= a^5.$$

Dividing by a^5, $a^0 = \frac{a^5}{a^5} = 1.$

That is:

Any number (not zero) with the exponent zero equals 1.

Thus, $5^0 = 1$, $10^0 = 1$, $(\frac{1}{2})^0 = 1$, $\left(\frac{a}{b}\right)^0 = 1$.

ORAL EXERCISES

State the value of:

1. $(ab)^0$.

2. $.9^0$.

3. 100^0.

4. $\left(\frac{a}{bc}\right)^0$.

5. $(-3)^0$.

6. $(\frac{1}{2})^0$.

7. $(-\frac{1}{27})^0$.

8. $3^2\cdot 3^0$.

9. $3^2\cdot 5^0$.

10. $\frac{1}{2}\cdot a^0$.

11. a^5a^0.

12. $a^m\cdot b^0$.

13. $a^5\div a^0$.

14. $4^3\div 4^0$.

15. $(\frac{1}{2})^0$.

16. $.9\div 100^0$.

17. $9\frac{1}{2}\cdot 5^0$.

18. $3^0\cdot 27^0$.

19. $(2^3\cdot 3^2)^0$.

127. The meaning of the negative exponent may be found as follows:

Assuming that Law I holds for negative exponents,

$$5^{-3} \cdot 5^{+3} = 5^{-3+3} = 5^0 = 1.$$

That is, 5^{-3} is a multiplier such that its product with 5^{+3} is 1. But if the product of two numbers is 1, one is the reciprocal of the other.

Therefore, 5^{-3} is the reciprocal of 5^3 which is $\frac{1}{5^3}$.

Expressed in general terms:

$$\begin{aligned} a^{-n} \cdot a^n &= a^{-n+n} \\ &= a^0 = 1. \\ a^{-n} &= \frac{1}{a^n}. \end{aligned}$$

In words:

a^{-n} *means* $\frac{1}{a^n}$, *for all values of* n, *positive or negative, integral or fractional.*

ORAL EXERCISES

Find similarly the meaning of:

1. 4^{-3}.
2. 2^{-4}.
3. $(\frac{3}{4})^{-2}$.
4. $(-5)^{-6}$.
5. a^{-3}.
6. $a^{-\frac{3}{4}}$.
7. $(\frac{1}{2})^{-\frac{2}{3}}$.
8. $(-\frac{2}{3})^{-\frac{4}{5}}$.

State the value of:

9. $4^{-\frac{1}{2}}$.
10. $8^{-\frac{1}{3}}$.
11. $8^{-\frac{2}{3}}$.
12. $(.2^0)^{\frac{1}{2}}$.
13. $16^{-\frac{1}{4}}$.
14. $16^{-\frac{3}{4}}$.
15. $.125^{-\frac{1}{3}}$.
16. $.125^{-\frac{2}{3}}$.
17. $32^{-\frac{2}{5}}$.
18. $27^{-\frac{2}{3}}$.
19. $.36^{-\frac{3}{2}}$.
20. $64^{-\frac{5}{6}}$.
21. a^{-3}.
22. $(a^0)^{-2}$.
23. $a^{-\frac{3}{5}}$.
24. $a^{-\frac{1}{3}}$.

WRITTEN EXERCISES

Perform the operations indicated, admitting negative exponents to the results:

1. $(4^2 \cdot 5^4)^{-\frac{1}{2}}$.
2. $2^{14} \cdot 2^{-6}$.
3. $2^{\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
4. $2^{-\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
5. $\frac{5^{-\frac{1}{2}} \cdot 5^{\frac{3}{2}}}{5^0}$.
6. $10^{-5} \cdot 10^{3} \cdot 10^{0}$.

7. $2^{-\frac{2}{3}} \cdot 2^{\frac{5}{6}}$.

8. $27^{-\frac{3}{4}} \cdot 9^{\frac{3}{2}}$.

9. $3^{\frac{7}{5}} \cdot 3^{-\frac{1}{2}} \cdot 4^0$.

10. $(x^{-4})^2$.

11. $10^7 \cdot 10^{-7}$.

12. $a^0 \cdot a^{\frac{1}{3}}$.

13. $\sqrt[4]{18^{-4}} \cdot \sqrt[3]{18^{-3}}$.

14. $\sqrt[3]{10^{-4}} \cdot \sqrt[5]{10^0}$.

15. $(10^{-4} \cdot 10^{-3})^{\frac{2}{7}}$.

16. $(10^{-\frac{2}{3}} \cdot 10^{\frac{1}{2}}$.

17. $10^{\frac{1}{2}} \cdot 10^{\frac{5}{6}} \cdot 10^{\frac{1}{3}} \cdot (\frac{4}{7})^0$.

18. $\dfrac{a^{-\frac{1}{2}}}{a^{-\frac{1}{3}}}$(or $a^{-\frac{1}{2}-(-\frac{1}{3})}$).

19. $a^{\frac{1}{2}} \cdot a^{-\frac{1}{2}}$(or $a^{\frac{1}{2}-\frac{1}{2}}$).

20. $\dfrac{a^{\frac{1}{2}}}{a^{-\frac{1}{2}}}$(or $a^{\frac{1}{2}-(-\frac{1}{2})}$).

21. $(a^{-\frac{1}{2}})^{-\frac{1}{3}}\,(x^{-4})^5$.

USE OF ZERO, NEGATIVE, AND FRACTIONAL EXPONENTS

128. We have defined zero and negative exponents so that Law I holds for them. It can be shown that the other four laws hold for these exponents as defined, but the laws will be applied here without proof.

129. The relation $a^{-n} \cdot a^n = a^0 = 1$ can be used to change the form of expressions.

I. **To free an expression from a negative exponent,** *multiply both numerator and denominator by a factor that will so combine with the factor having the negative exponent as to produce unity. If more than one negative exponent is involved, apply the process for each.*

For example:

$$7^{-3} \cdot 2^4 = \frac{7^3 \cdot 7^{-3} \cdot 2^4}{7^3} = \frac{2^4}{7^3}.$$

$$\frac{a^3}{b^{-5}} = \frac{b^5 \cdot a^3}{b^5 \cdot b^{-5}} = \frac{b^5a^3}{1} = b^5a^3$$

$$\frac{x^{-2}}{t^{-5}} = \frac{t^5x^2 \cdot x^{-2}}{t^5x^2 \cdot t^{-5}} = \frac{t^5}{x^2}.$$

WRITTEN EXERCISES

Write equivalent expressions without negative exponents:

1. $\dfrac{3^{-2}}{2^{-3}}$.

2. $\dfrac{a^{-1}b^{-2}}{x^{-4}}$.

3. $2\,a^{-\frac{1}{2}}$.

4. $\dfrac{ax^{-4}}{b^{-3}x^2}$.

5. $\dfrac{a^{-5}}{b^{-3}}$.
7. $\dfrac{x^{-6}}{a^{-2}}$.
9. $\dfrac{ab}{c^{-5}}$.
11. $\dfrac{5y^3}{5^{-1}y^{-4}}$.

6. $\dfrac{3^{-2}\cdot 4^{-1}}{5^{-3}}$.
8. $\dfrac{a^{-3}}{b^3}$.
10. $a^{-1}b^{-2}$.
12. $\dfrac{1}{a^{-2}x^{-3}}$.

II. **To free an expression from a fractional form,** *multiply both numerator and denominator by a factor that, in combination with the denominator, will produce unity. If more than one such form is involved, apply the process for each.*

For example:

$$\frac{a^3}{b^2}=\frac{b^{-2}a^3}{b^{-2}b^2}=b^{-2}a^3;$$

also,

$$\begin{aligned}\frac{2}{4^{-5}}+\frac{a^3}{x^2y^{-3}}&=\frac{4^5\cdot 2}{4^5\cdot 4^{-5}}+\frac{x^{-2}y^3a^3}{x^{-2}y^3x^2y^{-3}}\\&=4^5\cdot 2+\frac{x^{-2}y^3a^3}{(x^{-2}x^2)(y^3y^{-3})}\\&=2\cdot 4^5+a^3x^{-2}y^3.\end{aligned}$$

WRITTEN EXERCISES

Free from fractional forms:

1. $\dfrac{3}{5^2}$.
3. $\dfrac{a}{bx^3}$.
5. $\dfrac{1}{5^2}+\dfrac{3}{2^3}$.
7. $\dfrac{x}{y^{-1}}+\dfrac{a^2}{x^{-3}}$.

2. $\dfrac{1}{a^2b^3}$.
4. $\dfrac{a^{-2}}{t^{-3}}$.
6. $\dfrac{x^2}{y^3}+\dfrac{y^2}{x^3}$.
8. $\dfrac{a}{b}+\dfrac{a^{-1}}{b^{-1}}$.

III. **To transfer any specified factor from the numerator into the denominator, or vice versa,** *multiply the numerator and denominator by a factor that, in combination with the factor to be transferred, will produce unity.*

EXAMPLES

1. **Transferring the factors of the denominator to the numerator:**

$$\frac{x^7}{x^4y^{-3}}=\frac{x^{-4}y^3x^7}{x^{-4}y^3x^4y^{-3}}=x^{-4}y^3x^7=x^3y^3.$$

2. **Transferring the literal factors of the numerator to the denominator:**

$$\frac{5\,a^3b^2}{4\,a^4b^{-3}}=\frac{5\,a^{-3}b^{-2}a^3b^2}{4\,a^{-3}b^{-2}a^4b^{-3}}=\frac{5}{4\,ab^{-5}}.$$

WRITTEN EXERCISES

In the following expressions:

(*a*) Transfer all literal factors to the numerator.

(*b*) Transfer all literal factors to the denominator.

1. $\frac{a^2x^3}{a^4x^7}$.
2. $\frac{a^{\frac{3}{4}}}{b^{\frac{7}{8}}}$.
3. $\frac{6\,ay^2z^{-5}}{11\,a^3x^{-3}y^4}$.
4. $\frac{4\,a^2b^3c^5}{17\,a^3b^5c^{-3}}$.
5. $\frac{5\,a^{-\frac{1}{3}}b^{\frac{2}{3}}}{8\,a^{\frac{5}{6}}b^{-\frac{2}{3}}}$.
6. $\frac{a^{-\frac{2}{3}}b^{-5}c^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{-5}c^{-\frac{3}{4}}}$.
7. $\frac{5\,a^{-\frac{1}{3}}}{abx}$.
8. $\frac{6\,x^{-\frac{4}{5}}y^2}{5\,pq}$.

130. The laws of exponents enable us to perform operations with polynomials containing fractional and negative exponents.

Thus:

$$(a^{\frac{2}{3}} + b^{\frac{1}{2}})^2 = (a^{\frac{2}{3}})^2 + 2\,a^{\frac{2}{3}}b^{\frac{1}{2}} + (b^{\frac{1}{2}})^2$$
$$= a^{\frac{4}{3}} + 2\,a^{\frac{2}{3}}b^{\frac{1}{2}} + b.$$

WRITTEN EXERCISES

Perform the indicated operations, admitting negative exponents to the results:

1. $(a^{\frac{3}{2}} - b^{\frac{5}{4}})^2$.
2. $(4\,a^{\frac{2}{3}}x^{\frac{1}{2}} + 2^3)^2$.
3. $(a^{-\frac{1}{2}} + b^{\frac{3}{2}})^2$.
4. $(a^n - 3\,b^r)^2$.
5. $(x^{-\frac{3}{4}} - y^{-\frac{2}{3}})(x^{-\frac{3}{4}} + y^{-\frac{2}{3}})$.
6. $(x^{-n} + 1)(x^{-n} - 1)$.
7. $(x^{\frac{6}{5}} + 3)(x^{\frac{6}{5}} + 5)$.
8. $\frac{a^5 - x^4}{a^{\frac{5}{2}} - x^2}$.
9. $(x^n - y^n)^2$.
10. $(a^{2n}b^{3r} - 1)^2$.
11. $\frac{a^p \cdot b^q}{a^{p+1} \cdot b^{q-1}}$.
12. $(a^n + t^n)^2$.
13. $(a^{\frac{1}{3}}b^{\frac{3}{2}} + x^{-\frac{5}{2}})(a^{\frac{1}{3}}b^{\frac{3}{2}} - x^{-\frac{5}{2}})$.

Express as a product of two factors:

14. $x^6 - m^{-4}$.
15. $a^8 - 2\,a^4x^{\frac{1}{3}} + x^{\frac{2}{3}}$.
16. $a^{2n} + 2\,a^nb^n + b^{2n}$.
17. $x^{4m} - 4\,x^{2m}y^{2m} + 4\,y^{4m}$.
18. $y^{-7} - x^{-10}$.
19. $x^r - 4$.
20. $1 + 8\,x^{-\frac{5}{2}} + 16\,x^{-5}$.
21. $x^{12} + 6\,x^6y^{-\frac{7}{2}} + 9\,y^{-7}$.

EVOLUTION

Square Root of Binomials of the Form $a+\sqrt{b}$

131. Binomials of the form $a+\sqrt{b}$ can often be put into the form $x+y+2\sqrt{xy}$, or $(\sqrt{x}+\sqrt{y})^2$, and hence the square root, $\sqrt{x}+\sqrt{y}$, of the binomial can be written at once.

EXAMPLES

1. Find the square root of $4+2\sqrt{3}$.

$$4+2\sqrt{3}=3+1+2\sqrt{3\cdot 1}.$$

Hence, $x+y=3+1$ and $xy=3\cdot 1$, from which $x=3$ and $y=1$.

$$\therefore \sqrt{4+2\sqrt{3}}=\pm(\sqrt{3}+\sqrt{1})=\pm(\sqrt{3}+1).$$

The coefficient of the radical must be made 2 in order to apply the formula $x+y+2\sqrt{xy}$.

2. Find the square root of $3-\sqrt{8}$.

$$3-\sqrt{8}=3-\sqrt{4\cdot 2}=3-2\sqrt{2}.$$

$$\therefore x+y=3, \text{ and } xy=2. \quad \therefore x=2,\ y=1 \text{ by inspection.}$$

$$\therefore \sqrt{3-\sqrt{8}}=\pm(\sqrt{2}-\sqrt{1})=\pm(\sqrt{2}-1).$$

3. Find the square root of $7+4\sqrt{3}$.

$$7+4\sqrt{3}=7+2\sqrt{4\cdot 3};\ x+y=7,\ xy=12;\ \therefore x=4,\ y=3.$$

$$\therefore \sqrt{7+4\sqrt{3}}=\pm(\sqrt{4}+\sqrt{3})=\pm(2+\sqrt{3}).$$

The square root as a whole may be taken positively or negatively, as in the case of rational roots.

The solution of these problems depends upon finding two numbers whose sum and product are given. This can sometimes be done by inspection, but the general problem is one of simultaneous equations.

WRITTEN EXERCISES

Find the square root of:

1. $11+6\sqrt{2}$.
2. $8-2\sqrt{15}$.
3. $49-12\sqrt{10}$.
4. $41-24\sqrt{2}$.
5. $2\frac{1}{4}-\sqrt{5}$.
6. $2\frac{1}{3}-\frac{4}{3}\sqrt{3}$.
7. $17+12\sqrt{2}$.
8. $\frac{3}{2}\sqrt{5}+3\frac{1}{2}$.
9. $56-24\sqrt{5}$.

Cube Root of Arithmetical Numbers

132. Pointing off into Periods. Since $10^3 = 1000$, we know that the cube root of any number greater than 1 but less than 1000 is less than 10. Its integral part consists of one figure.

Since $\overline{100}^3 = 1{,}000{,}000$, we know that the cube root of any number greater than 1000 but less than 1,000,000 is greater than 10 but less than 100. That is, if the given number has from 4 to 6 digits in its integral part, its cube root will have 2 digits in its integral part. If larger numbers are given, the above reasoning can be repeated for 1000^3, etc., showing that in all cases if the number be pointed off into periods of 3 digits each (or possibly fewer in the left period), then each period will correspond to a digit of the root.

133. The cube root of the left period can be found approximately by inspection, and the number so found, with a zero annexed for each other period, will be an approximate value for the root.

EXAMPLE

Find $\sqrt[3]{481890304}$.

Pointing off as above:

$$481'890'304.$$

By trial we find $7^3 = 343$, and $8^3 = 512$. Hence 700 is an approximate value for the root. It may be verified that $\overline{700}^3$ is less than the given number, and that $\overline{800}^3$ is more than the given number. That is, the hundreds' figure of the root is 7.

WRITTEN EXERCISES

Find, as above, the first figure of:

1. $\sqrt[3]{493039}$.
2. $\sqrt[3]{2924207}$.
3. $\sqrt[3]{57066625}$.
4. $\sqrt[3]{254840104}$.
5. $\sqrt[3]{1345572864}$.
6. $\sqrt[3]{62287505344}$.

134. When once an approximate value a has been found for the root, an approximate value for the remainder, r, of the root can be found by means of the formula:

$$(a+r)^3 = a^3 + 3a^2r + 3ar^2 + r^3.$$

For example:

In $\sqrt[3]{238328}$ we find as above that $a = 60$.

$$\begin{array}{lrl} \text{Then,} & a^3 + 3a^2r + 3ar^2 + r^3 = & 238'328, \text{ the whole cube} \\ \text{Subtracting} & a^3 \qquad\qquad\qquad\quad = & 216000, \text{ the cube of the part found,} \\ \hline & 3a^2r + 3ar^2 + r^3 = & 22'328, \text{ the first remainder.} \end{array}$$

Since something must be added to $3a^2r$ to make it equal to 22,328,

$$3a^2r \text{ is less than } 22'328,$$

or

$$r \text{ is less than } \frac{22'328}{3a^2}.$$

Consequently, the first figure of this quotient will either be the first figure of r or greater than it. In this instance $3a^2$ is 10,800, hence the first figure of the quotient is 2.

Trying 2 as r, we have to calculate $3a^2r + 3ar^2 + r^3$. This is most conveniently done by using the form $(3a^2 + 3ar + r^2)r$.

$$\begin{array}{lr} \text{We have already} & 3a^2 = 10800. \\ \text{We find:} & 3ar = \quad 360 \\ & r^2 = \qquad 4 \\ \hline \text{Adding:} & 3a^2 + 3ar + r^2 = 11164 \\ \text{Then,} & r(3a^2 + 3ar + r^2) = 22328 \end{array}$$

The calculation should be arranged thus:

$$\begin{array}{lr|l} & & \underline{6 \quad 2} \\ & & 238'328 \\ & a^3 = & \underline{216\,000} \\ \text{Trial divisor:} & 3a^2 = 10800 & 22328 \\ & 3ar = 3 \cdot 2 \cdot 60 = \quad 360 & \\ & r^2 = 2^2 = \qquad 4 & \\ \hline \text{Complete divisor:} & 11164 & \underline{22328 \text{ or } 2 \times 11164} \end{array}$$

If the root consists of more than two figures, the above work is repeated, using the part of the root already found as a.

If it should happen that the product of r and the complete divisor is larger than the remainder of the number, try the next smaller digit for r.

When necessary, periods are pointed off to the right of the decimal point.

WRITTEN EXERCISES

Find:

1. $\sqrt[3]{74088}$.
2. $\sqrt[3]{148877}$.
3. $\sqrt[3]{2803221}$.
4. $\sqrt[3]{16387064}$.
5. $\sqrt[3]{55306341}$.
6. $\sqrt[3]{143055667}$

Find these roots to one decimal place:

7. $\sqrt[3]{637}$. 8. $\sqrt[3]{3485}$. 9. $\sqrt[3]{263488}$.

Find these roots to two decimal places:

10. $\sqrt[3]{5}$. 11. $\sqrt[3]{17}$. 12. $\sqrt[3]{269}$. 13. $\sqrt[3]{4.763}$.

Cube Root of Polynomials

135. The *cube roots* of polynomials may be extracted in a similar manner:

EXAMPLE

Extract the cube root of $27x^3 - 27x^2y + 9xy^2 - y^3$.

$$
\begin{array}{lr|r}
 & \text{Root} & \underline{3x - y} \\
a^3 + 3a^2r + 3ar^2 + r^3 = & \text{Power} & 27x^3 - 27x^2y + 9xy^2 - y^3 \\
a = 3x & a^3 = (3x)^3 = & \underline{27x^3 \qquad\qquad\qquad\qquad} \\
\text{Trial divisor} = 27x^2 \quad \therefore r = -y & & -27x^2y + 9xy^2 - y^3 \\
\text{Complete divisor} = [3(3x)^2 - 3(3x)y + y^2] & & \underline{-27x^2y + 9xy^2 - y^3}
\end{array}
$$

EXPLANATION.

1. Arrange the expression in the order of the powers of some letter, as x.

2. Take the cube root of the first term of the power for the first term of the root, as $3x$.

3. Divide the second term of the power by 3 times the square of the first term of the root. The result is the second term of the root, as $-y$.

4. If a denote the approximate value of the root already found ($3x$ in the above instance), and r the value to be used as the next term ($-y$ in the above instance), form the complete divisor $3a^2 + 3ar + r^2$, multiply it by r, and subtract.

5. If there is a further remainder, proceed as before, using the entire part of the root already found as a.

WRITTEN EXERCISES

Extract the cube root of:

1. $8x^3 + 12x^2 + 6x + 1$.
2. $27 - 27x + 9x^2 - x^3$.
3. $8a^3b^3 - 12a^2b^2 + 6ab - 1$.
4. $x^3 - 3x^2y + 3xy^2 - y^3$.
5. $8x^6 - 12x^4 + 6x^2 - 1$.
6. $x^{12} - 3x^8m + 3x^4m^2 - m^3$.

7. $-a^6-6a^5+40a^3-96a+64.$

8. $54x^2y^2-36x^4y+8x^6-27y^3.$

9. $64x^6-144x^5+156x^4-99x^3+39x^2-9x+1.$

10. $-64+96x-40x^3+6x^5+x^6.$

11. $27y^3-27y^2+63y-37+\frac{42}{y}-\frac{12}{y^2}+\frac{8}{y^3}.$

12. $8x^3-12x+\frac{6}{x}-\frac{1}{x^3}.$

13. $a^3-b^3+c^3+3ab^2+3ac^2-3bc^2+3a^2c+3b^2c$
$-3a^2b-6abc.$

14. $27x^3-27x+\frac{9}{x}-\frac{1}{x^3}.$

15. $64(ab)^{3p}-48(ab)^{2p}+12(ab)^p-1.$

16. $30a^{-1}+8a^{-3}+8a^3+30a-12a^2-25-12a^{-2}.$

17. $a^6-20a^3-6a+15a^4-6a^5+15a^2+1.$

18. $x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}.$

19. $x^{3p}-y^{3q}-z^{3r}-3x^{2p}y^q+3x^py^{2q}-3x^{2p}z^r+3x^pz^{2r}-3y^{2q}z^r$
$-3y^qz^{2r}+6x^py^qz^r.$

20. $27x^3+y^3-\frac{1}{z^3}+9xy^2+\frac{9x}{z^2}+27x^2y-27\frac{x^2}{z}-18\frac{xy}{z}$
$+\frac{3y}{z^2}-\frac{3y^2}{z}.$

REVIEW

WRITTEN EXERCISES

Express with positive exponents:

1. $m^{-\frac{3}{4}}n^8.$ 2. $4x^{-\frac{1}{2}}y^{-1}z.$ 3. $3a^{-5}b^5.$ 4. $17x^{-\frac{5}{2}}y^{-7}z^{-\frac{7}{8}}.$

Transfer all literal factors from the denominator to the numerator:

5. $\frac{x^{\frac{3}{4}}}{x^{-\frac{2}{3}}}.$ 6. $\frac{ab}{a^{-\frac{1}{2}}b^{-3}}.$ 7. $\frac{1}{6x^{-2}y^{\frac{3}{2}}}.$ 8. $\frac{5}{ay^{-4}}.$

Multiply:

9. $(2+\sqrt{x+1})^2$.

10. $\sqrt{5}\cdot\sqrt[3]{6}$.

11. $5\sqrt[4]{m^{-3}}\cdot 2\,m^{-1}$.

12. $p\cdot p^{-\frac{3}{4}}$.

13. $(a^{-1}-b^{-1})(a^{-\frac{1}{2}}-b^{-\frac{1}{2}})$.

14. $(x^2-1)(x^{\frac{1}{2}}+1)$.

Remove the parentheses:

15. $(a^{-\frac{3}{4}})^2$.

16. $(\sqrt[3]{a^{-1}})^3$.

17. $\left(\frac{1}{\sqrt[4]{p^3}}\right)^{\frac{4}{3}}$.

18. $\left(\frac{1}{n^{-2}}\right)^5$.

19. $\left(a^{\frac{5n}{6}}\right)^{\frac{3}{10n}}$.

20. $\left(x^{\frac{m^2}{n^2}-1}\right)^{\frac{n}{m-n}}$.

21. $\left(\frac{1}{x^{-\frac{p}{q}}}\right)^{\frac{q}{p}}$.

22. $\left(m^{\frac{pq}{xy}}\right)^{\frac{x^2y}{pq}}$.

23. $\left(s^{\frac{1}{n-1}}\right)^{n^2-1}$.

Simplify as far as possible, admitting negative exponents to the results:

24. $(8\,a^3b^{\frac{1}{2}}c^{-\frac{3}{4}})^{\frac{2}{3}}$.

25. $(16\,a^{\frac{2}{3}}bc^{-4})^{\frac{3}{4}}$.

26. $(5\,ax^3y^2)^{\frac{1}{2}}$.

27. $(\sqrt[3]{4\,x^2})^{\frac{1}{4}}$.

28. $(64\,x^{-3}y^{\frac{1}{2}})^{-\frac{1}{6}}$.

29. $(-250\,x^3y^{-\frac{3}{5}}z^{-3})^{\frac{2}{3}}$.

30. $(-243\,a^{-\frac{1}{2}}y^{-1}z^{\frac{1}{4}})^{-\frac{4}{5}}$.

31. $\sqrt[p-q]{x^{p^2-q^2}}$.

32. $\sqrt[p]{x^{-2p}y^{-p^2}z^{-p}}$.

33. $\sqrt{a^{-m}b^{4m}}$.

34. $3\,a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-1}\cdot 2\,a^{\frac{1}{4}}b^{\frac{1}{2}}c$.

35. $10\,ab^{-\frac{1}{2}}c^{\frac{3}{4}}\div 2\,a^{-1}b^2z^{\frac{1}{2}}$.

36. $a^mb^{-2n}c^{-2}\div ab^nz^{-3}$.

37. $(3\,a^{-2}\div b^{-2})^{-5}$.

38. $(a^{p-q})^{p+q}\cdot a^{p^2}a^{q^2}$.

39. $\frac{\sqrt{a}}{\sqrt[5]{a^2b^3}}\cdot\frac{\sqrt[4]{a^3b^2}}{\sqrt[10]{a^7b^9}}$.

40. $x^{-3}\cdot x^{\frac{3}{4}}\cdot x^2\cdot\sqrt[5]{x}$.

41. $\frac{\sqrt[3]{x}}{\sqrt{x}}\cdot\frac{x^3}{x}\cdot\frac{\sqrt[3]{x^2}}{x^2}$.

42. $[\{(a^2-b^{-2})^{-1}\}^{-2}]^5$.

43. $[(a^{-\frac{1}{2}})^{\frac{2}{3}}]^{-12}$.

44. $\left\{\sqrt{ab^{-2}\sqrt{ab}}\right\}^4$.

45. $\{(a^{-3}b^2)^{\frac{1}{2}}\}^{-\frac{2}{3}}$.

46. $\sqrt[3]{a^2\sqrt{a^{-1}}}$.

47. $[(x^a)^{-b}]^{-\frac{1}{a}}\div[(x^{-b})^c]^{-\frac{1}{c}}$.

48. $-a^3b^{-4}c^{-3}d^5\cdot -a^{-2}b^5c^4d^{-5}$.

7. $-a^6-6a^5+40a^3-96a+64.$

8. $54x^2y^2-36x^4y+8x^6-27y^3.$

9. $64x^6-144x^5+156x^4-99x^3+39x^2-9x+1.$

10. $-64+96x-40x^3+6x^5+x^6.$

11. $27y^3-27y^2+63y-37+\frac{42}{y}-\frac{12}{y^2}+\frac{8}{y^3}.$

12. $8x^3-12x+\frac{6}{x}-\frac{1}{x^3}.$

13. $a^3-b^3+c^3+3ab^2+3ac^2-3bc^2+3a^2c+3b^2c$
$-3a^2b-6abc.$

14. $27x^3-27x+\frac{9}{x}-\frac{1}{x^3}.$

15. $64(ab)^{3p}-48(ab)^{2p}+12(ab)^p-1.$

16. $30a^{-1}+8a^{-3}+8a^3+30a-12a^2-25-12a^{-2}.$

17. $a^6-20a^3-6a+15a^4-6a^5+15a^2+1.$

18. $x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}.$

19. $x^{3p}-y^{3q}-z^{3r}-3x^{2p}y^q+3x^py^{2q}-3x^{2p}z^r+3x^pz^{2r}-3y^{2q}z^r$
$-3y^qz^{2r}+6x^py^qz^r.$

20. $27x^3+y^3-\frac{1}{z^3}+9xy^2+\frac{9x}{z^2}+27x^2y-27\frac{x^2}{z}-18\frac{xy}{z}$
$+\frac{3y}{z^2}-\frac{3y^2}{z}.$

REVIEW

WRITTEN EXERCISES

Express with positive exponents:

1. $m^{-\frac{3}{4}}n^8.$ 2. $4x^{-\frac{1}{2}}y^{-1}z.$ 3. $3a^{-5}b^5.$ 4. $17x^{-\frac{5}{2}}y^{-7}z^{-\frac{7}{8}}.$

Transfer all literal factors from the denominator to the numerator:

5. $\frac{x^{\frac{3}{4}}}{x^{-\frac{2}{3}}}.$ 6. $\frac{ab}{a^{-\frac{1}{7}}b^{-3}}.$ 7. $\frac{1}{6x^{-2}y^{\frac{3}{2}}}.$ 8. $\frac{5}{ay^{-4}}.$

Multiply:

9. $(2+\sqrt{x+1})^2$.

10. $\sqrt{5}\cdot\sqrt[3]{6}$.

11. $5\sqrt[4]{m^{-3}}\cdot 2\,m^{-1}$.

12. $p\cdot p^{-\frac{3}{4}}$.

13. $(a^{-1}-b^{-1})(a^{-\frac{1}{2}}-b^{-\frac{1}{2}})$.

14. $(x^2-1)(x^{\frac{1}{2}}+1)$.

Remove the parentheses:

15. $(a^{-\frac{3}{4}})^2$.

16. $(\sqrt[3]{a^{-1}})^3$.

17. $\left(\frac{1}{\sqrt[4]{p^3}}\right)^{\frac{4}{3}}$.

18. $\left(\frac{1}{n^{-\frac{2}{5}}}\right)^5$.

19. $\left(a^{\frac{5n}{6}}\right)^{\frac{3}{10n}}$.

20. $\left(x^{\frac{m^2}{n^2}-1}\right)^{\frac{n}{m-n}}$.

21. $\left(\frac{1}{x^{-\frac{p}{q}}}\right)^{\frac{q}{p}}$.

22. $\left(m^{\frac{pq}{xy}}\right)^{\frac{x^2y}{pq}}$.

23. $\left(s^{\frac{1}{n-1}}\right)^{n^2-1}$.

Simplify as far as possible, admitting negative exponents to the results:

24. $(8\,a^3b^{\frac{1}{2}}c^{-\frac{3}{4}})^{\frac{2}{3}}$.

25. $(16\,a^{\frac{2}{3}}bc^{-4})^{\frac{3}{4}}$.

26. $(5\,ax^3y^2)^{\frac{1}{2}}$.

27. $(\sqrt[3]{4\,x^2})^{\frac{1}{4}}$.

28. $(64\,x^{-3}y^{\frac{1}{2}})^{-\frac{1}{6}}$.

29. $(-250\,x^3y^{-\frac{3}{5}}z^{-3})^{\frac{2}{3}}$.

30. $(-243\,a^{-\frac{1}{2}}y^{-1}z^{\frac{1}{4}})^{-\frac{4}{5}}$.

31. $\sqrt[p-q]{x^{p^2-q^2}}$.

32. $\sqrt[p]{x^{-2p}y^{-p^2}z^{-p}}$.

33. $\sqrt{a^{-m}b^{4m}}$.

34. $3\,a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-1}\cdot 2\,a^{\frac{1}{4}}b^{\frac{1}{2}}c$.

35. $10\,ab^{-\frac{1}{2}}c^{\frac{3}{4}}\div 2\,a^{-1}b^2z^{\frac{1}{2}}$.

36. $a^mb^{-2n}c^{-2}\div ab^nz^{-3}$.

37. $(3\,a^{-2}\div b^{-2})^{-5}$.

38. $(a^{p-q})^{p+q}\cdot a^{p^2}a^{q^2}$.

39. $\frac{\sqrt{a}}{\sqrt[5]{a^2b^3}}\cdot\frac{\sqrt[4]{a^3b^2}}{\sqrt[10]{a^7b^9}}$.

40. $x^{-3}\cdot x^{\frac{3}{4}}\cdot x^2\cdot\sqrt[5]{x}$.

41. $\frac{\sqrt[3]{x}}{\sqrt{x}}\cdot\frac{x^3}{x}\cdot\frac{\sqrt[3]{x^2}}{x^2}$.

42. $[\{(a^2-b^{-2})^{-1}\}^{-2}]^5$.

43. $[(a^{-\frac{1}{2}})^{\frac{2}{3}}]^{-12}$.

44. $\left\{\sqrt{ab^{-2}\sqrt{ab}}\right\}^4$.

45. $\{(a^{-3}b^2)^{\frac{1}{2}}\}^{-\frac{2}{3}}$.

46. $\sqrt[3]{a^2\sqrt{a^{-1}}}$.

47. $[(x^a)^{-b}]^{-\frac{1}{a}}\div[(x^{-b})^c]^{-\frac{1}{c}}$.

48. $-a^3b^{-4}c^{-3}d^5\cdot -a^{-2}b^5c^4d^{-5}$.

49. $a^2x - 3\,y^6 \cdot a^3x^6y^9$.

50. $3\,x^{-\frac{1}{2}}5\,x^{\frac{4}{3}} \cdot 10\,x^{-\frac{1}{4}}$.

51. $\sqrt{a^{-1}\sqrt{a^3\sqrt{a-4}}}$.

52. $\{x^{-\frac{3}{2}}y(xy^{-2})^{-\frac{1}{2}}(x^{-1}y)^{\frac{2}{3}}\}^{\frac{3}{2}}$.

53. $\dfrac{\dfrac{x-y}{2y+x}+\dfrac{1}{2}}{3\frac{1}{2}-\left(\dfrac{x+2\,y}{5\,x+7\,y}\right)^{-1}}$.

54. $\left(x^{\frac{n+1}{n-1}} \div x^{\frac{n-1}{n+1}}\right)^{\frac{n-1}{2n}}$.

55. $[(a^{p+q})^{p-q}(a^{q^3})^{\frac{1}{q}}]^{\frac{1}{p^2}}$.

56. $\dfrac{\sqrt{(2^{\frac{1}{3}}+2^{-\frac{1}{3}})^2-4}}{4\,(2^{\frac{1}{3}}-2^{-\frac{1}{3}})} \cdot \dfrac{5+\sqrt{21}}{5-\sqrt{21}}$.

57. Add: $\frac{1}{3}\sqrt{45}+4\sqrt{\frac{5}{4}}-\sqrt{125}$.

58. Express the product $\sqrt[3]{a^2}\sqrt{a^3}$ as a single surd.

59. Divide $2\,x^{\frac{9}{2}}y^{-3}-5\,x^{\frac{7}{2}}y^{-2}+7\,x^{\frac{5}{2}}y^{-1}-5\,x^{\frac{3}{2}}+2\,x^{\frac{1}{2}}y$ by $x^3y^{-3}-x^2y^{-2}+xy^{-1}$.

Find equivalent expressions with rational divisors:

60. $3\sqrt{2\,a} \div 2\sqrt{3\,b}$.

61. $b\sqrt{a^2} \div \sqrt{ab}$.

62. $\sqrt{40\,x^3y} \div x\sqrt{5\,y}$.

63. $x\sqrt[3]{y} \div y\sqrt[3]{x}$.

64. $2\sqrt[3]{2\,a^2} \div \sqrt[3]{4\,a}$.

65. $-\frac{3}{4}\sqrt{\frac{3}{5}} \div \frac{3}{10}\sqrt{3}$.

66. $6\sqrt[3]{54\,x^2} \div 2\sqrt[3]{2\,x^2}$.

67 $a^2\sqrt[4]{48\,ab^3} \div 2\,ab\sqrt[4]{3\,ab^2}$.

68. $4\,ax \div \sqrt[4]{ax}$.

69. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$.

70. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$.

71. $\dfrac{\sqrt{x}-\sqrt{x+y}}{\sqrt{x}+\sqrt{x+y}}$.

72. $\dfrac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}-b}$.

73. $\dfrac{3\sqrt{x-3}+\sqrt{x+3}}{3\sqrt{x-3}-\sqrt{x+3}}$.

74. $\dfrac{1}{a-\sqrt{a^2-x^2}}$.

75. $\dfrac{p-\sqrt{q}}{p+\sqrt{q}}$.

76. $\dfrac{1}{2+\sqrt{5}-\sqrt{2}}$.

77. $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$.

78. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$.

Solve:

79. $3x + \sqrt{x^2 - 2x + 5} = 1.$ 80. $\dfrac{\sqrt{x} + a}{b} = \dfrac{x}{a^2 - 2ab + b^2}.$

81. $\dfrac{x}{a+b} + \sqrt{(a+b)^2 + ab - x} = a + b + \dfrac{2ab}{a+b}.$

82. $\sqrt{x}\,(2a - b + \sqrt{x}) = 3a^2 - ab.$

Extract the square root of:

83. $7 - 2\sqrt{10}.$ 84. $a + 2b + \sqrt{8ab}.$

Extract the cube root of:

85. $8x^3 - 12x^2y^{\frac{1}{3}} + 6xy^{\frac{2}{3}} - y.$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Explain the meaning of the exponent in $a^{\frac{p}{q}}$. Sec. 118.

2. Explain the meaning of the exponent zero. Sec. 126.

3. Explain the meaning of the exponent in a^{-n}. Sec. 127.

4. State the *Law of Exponents* for multiplication; for division; for involution. Secs. 110–113.

5. How are expressions multiplied whose bases are different and whose exponents are different? Secs. 124, 128.

6. How may an expression with negative exponents be changed to an equal one with positive exponents? Sec. 129, I.

7. How free an expression from the fractional form by the use of exponents? Sec. 129, II.

8. How may a factor be transferred from one term of a fraction to the other by use of an exponent? Sec. 129, III.

9. How may the square root of a binomial surd be found by inspection? Sec. 131.

10. What formula may be used to extract the cube root of a number or of an algebraic expression? Sec. 134.

HISTORICAL NOTE

The use of exponents to denote powers and roots seem so essential to our present-day work in algebra that it is difficult to imagine how algebra could exist without this notation; but their general use is comparatively recent. Vieta (1591), the founder of modern algebra, had no knowledge of them. In place of our notation he used a system much like that of Diophantos. Thus, for the expression

$$x^3 - 2x^2 + 5\ x - 3 = 7,$$

Vieta wrote $1\,C - 2\,Q + 5\,N - 3\ \text{aequ.}\ 7,$

in which C, Q, and N are the first letters of the Latin words meaning "cube," "square," and "number." By number is meant the unknown quantity, and by C and Q are meant the square and cube of this number. In the sixteenth century Stifel used integral exponents, and Stevin invented fractional exponents, if we pass over the beginnings made by Oresme (1382), but it remained for Wallis and Newton, in the seventeenth century, to popularize these improvements.

John Wallis, born in 1616, was second only to Newton among English mathematicians of the seventeenth century, and became a professor of geometry at Oxford College at the age of 33. He was a charter member of the famous Royal Society of Great Britain, founded in 1663.

JOHN WALLIS

Exponents, like the binomial theorem and many other principles of algebra, were standardized and made popular through their application to concrete problems, for Wallis, in seeking the areas inclosed by various curves, used the series of powers, x^3, x^2, x^1, x^0, x^{-1}, x^{-2}, x^{-3} in its present meaning, thus placing upon these symbols his stamp of authority. In this way our present definitions of negative exponents and of the zero exponent were established. The fact that these interpretations satisfy the fundamental laws of algebra was shown by Peacock and Hamilton, as noted (p. 36).

CHAPTER IV

LOGARITHMS

MEANING AND USE OF LOGARITHMS

136. Use of Exponents in Computation. By applying the laws of exponents certain mathematical operations may be performed by means of simpler ones. The following table of powers of 2 may be used in illustrating some of these simplifications:

$1 = 2^0$
$2 = 2^1$
$4 = 2^2$
$8 = 2^3$
$16 = 2^4$

$32 = 2^5$
$64 = 2^6$
$128 = 2^7$
$256 = 2^8$
$512 = 2^9$

$1024 = 2^{10}$
$2048 = 2^{11}$
$4096 = 2^{12}$
$8192 = 2^{13}$
$16384 = 2^{14}$

$32768 = 2^{15}$
$65536 = 2^{16}$
$131072 = 2^{17}$
$262144 = 2^{18}$
$524288 = 2^{19}$

137. Application of Law I, Section 110, p. 65.

EXAMPLES

1. Find: $8 \cdot 32$.

From the table, $8 = 2^3$, *(1)*
and $32 = 2^5$. *(2)*
Then, $8 \cdot 32 = 2^3 \cdot 2^5 = 2^8$, *(3*
and, according to the table, $2^8 = 256$. *(4)*

2. Find: $2048 \cdot 64$.

From the table, $2048 = 2^{11}$, *(1)*
and $64 = 2^6$. *(2)*
Then, $2048 \cdot 64 = 2^{11} \cdot 2^6 = 2^{17}$, *(3)*
and, according to the table, $2^{17} = 131072$. *(4)*

Thus, the process is simply one of inspection. In the above example we merely added 11 and 6 and looked in the table for the number opposite to 2^{17}.

ORAL EXERCISES

State the following products by reference to the table:

1. $16 \cdot 256$.
2. $32 \cdot 128$.
3. $64 \cdot 512$.
4. $8 \cdot 2048$.
5. $32 \cdot 32$.
6. $64 \cdot 64$.
7. $32 \cdot 2048$.
8. $16 \cdot 4096$.
9. $128 \cdot 512$.
10. $128 \cdot 1024$.
11. $8 \cdot 16384$.
12. $32 \cdot 4096$.

138. Application of Law II, Section 111, p. 66.

EXAMPLES

1. Find: $\frac{256}{32}$.

From the table, $256 = 2^8$, *(1)*

and $32 = 2^5$. *(2)*

Hence, $\frac{256}{32} = \frac{2^8}{2^5} = 2^{8-5} = 2^3$, *(3)*

and, according to the table, $2^3 = 8$. *(4)*

2. Find: $\frac{65536}{2048}$.

As above, $\frac{65536}{2048} = \frac{2^{16}}{2^{11}} = 2^5 = 32$.

ORAL EXERCISES

By use of the table determine the value of the following:

1. $\frac{1024}{128}$.
2. $\frac{8192}{64}$.
3. $\frac{32768}{1024}$.
4. $\frac{64 \cdot 512}{16 \cdot 128}$.
5. $\frac{32 \cdot 2048}{512}$.
6. $\frac{128 \cdot 131072}{64 \cdot 8 \cdot 8192}$.

139. Application of Law III, Section 112, p. 66.

EXAMPLES

1. Find: 16^3.

By the table, $16 = 2^4$. *(1)*

Hence, $16^3 = (2^4)^3 = 2^{12}$. *(2)*

and, according to the table, $2^{12} = 4096$. *(3)*

2. Find: $\sqrt{1024}$.

As above, $1024 = (1024)^{\frac{1}{2}} = (2^{10})^{\frac{1}{2}} = 2^5 = 32$, according to the table.

3. Find: $\sqrt[5]{32768}$.

As above, $\sqrt[5]{32768} = (2^{15})^{\frac{1}{5}} = 2^3 = 8$.

ORAL EXERCISES

By use of the table find the value of:

1. 32^3.
2. 3^5.
3. 32^5.
4. 64^2.
5. 256^2.
6. 16^4.
7. $\sqrt{8192}$.
8. $\sqrt[3]{4096}$.
9. $\sqrt[4]{65536}$.
10. $512^{\frac{2}{3}}$.
11. $\sqrt[3]{32768}$.
12. $\sqrt[5]{1024}$.

140. The examples and exercises above show that the laws of exponents furnish a powerful and remarkably easy way of making certain computations.

In the above illustrations we have used a table based on the number 2, and have limited the table to integral exponents; but for practical purposes a table based on 10 is used and is made to include fractional exponents.

For example:

1. It is known that approximately,

$$2 = 10^{\frac{3}{10}} \text{ or } 10^{.3} \text{ (more accurately } 10^{.301}\text{).}$$

From this we can express 20 as a power of 10, for

$$20 = 10 \cdot 2 = 10^1 \cdot 10^{.301} = 10^{1.301}.$$

Similarly, $$200 = 10 \cdot 20 = 10^1 \cdot 10^{1.301} = 10^{2.301},$$

and $$2000 = 10 \cdot 200 = 10^1 \cdot 10^{2.301} = 10^{3.301}.$$

2. It is known that approximately $763 = 10^{2.88}$.

Then $$7630 = 10 \cdot 763 = 10^1 \cdot 10^{2.88} = 10^{3.88},$$

and $$76300 = 100 \cdot 763 = 10^2 \cdot 10^{2.88} = 10^{4.88}.$$

Similarly, $$76.3 = \frac{763}{10} = \frac{10^{2.88}}{10^1} = 10^{2.88-1} = 10^{1.88},$$

and $$7.63 = \frac{763}{100} = \frac{10^{2.88}}{10^2} = 10^{2.88-2} = 10^{0.88}.$$

WRITTEN EXERCISES

Given $48 = 10^{1.68}$; express as a power of 10:

1. 480. 2. 4800. 3. 48,000. 4. 4.8.

Given $649 = 10^{2.81}$; express as a power of 10:

5. 6490. 7. 649,000. 9. 6.49.
6. 64,900. 8. 64.9. 10. 649,000,000.

Given $300 = 10^{2.47}$; express as a power of 10:

11. 3. 13. 3000. 15. 300,000.
12. 30. 14. 30,000. 16. 3,000,000.

141. The use of the base 10 has several advantages.

I. The exponents of numbers not in the table can readily be found by means of the table.

To make this clear, let us suppose that a certain table expresses all integers from 100 to 999 as powers of 10; then 30, although not in this table, can be expressed as a power of 10 by reference to the table.

For, $30 = \frac{300}{10}$, and since 300 is in the supposed table we may find by reference to the table that $300 = 10^{2.47}$, and hence, $30 = \frac{10^{2.47}}{10^1} = 10^{1.47}$.

Similarly, 3.76 is not in the supposed table, but 376 is and $3.76 = \frac{376}{100} = \frac{376}{10^2}$. Therefore it is necessary only to subtract 2 from the power of 10 found for 376 in order to find the power of 10 equal to 3.76.

Similarly, 4680 is not in the table, but 468 is, and $4680 = 468 \cdot 10^1$. Therefore it is necessary only to add 1 to the power of 10 found for 468 in order to find the power of 10 equal to 4680.

Such a table would not enable us to express in powers of 10 numbers like 4683, 46.83, and 356,900, but only numbers of 3 or fewer digits, which may be followed by any number of zeros.

Similar conditions would apply to a table of powers for numbers from 1000 to 9999, from 10,000 to 99,999, and so on.

II. The integral part of the exponent can be written without reference to a table.

For example:

1. 879 is greater than 100, which is the second power of 10, and less than 1000, or the third power of 10. That is, 879 is greater than 10^2 but less than 10^3. Therefore the exponent of the power of 10 which equals 879 is 2. + a decimal.

2. Similarly, 87.9 lies between 10 and 100, or between 10^1 and 10^2, hence the exponent of the power of 10 that is equal to 87.9 is 1. + a decimal.

ORAL EXERCISES

State the integral part of the exponent of the power of 10 equal to each of the following:

1. 35.
2. 350.
3. 36.5.
4. 25.
5. 2500.
6. 36,500.
7. 25.5.
8. 365.5.
9. 17.65.

III. If two numbers have the same sequence of digits but differ in the position of the decimal point, the exponents of the powers of 10 which they equal have the same decimal part.

For example:

Given that $274.3 = 10^{2.43}$,

we have $27.43 = \frac{274.3}{10} = \frac{10^{2.43}}{10^1} = 10^{1.43}$,

also $2743 = 10 \cdot 274.3 = 10^1 \cdot 10^{2.43} = 10^{3.43}$,

also $274,300 = 1000 \cdot 274.3 = 10^3 \cdot 10^{2.43} = 10^{5.43}$.

In each instance the decimal part of the exponent is the same. It is evident that this will be the case in all similar instances, for shifting the decimal point is equivalent to multiplying or dividing repeatedly by 10, which is equivalent to changing the integral part of the exponent by adding or subtracting an integer.

ORAL EXERCISES

Given $647 = 10^{2.81}$, state the decimal part of the exponent of the power of 10 that equals:

1. 64.7.
2. 6.47.
3. 6470.
4. 64,700.
5. 647,000.
6. 6,470,000.

Given $568.1 = 10^{2.75}$, state the decimal part of the exponent of the power of 10 that equals:

7. 56.81.
9. 5681.
11. 568,100.
8. 5.681.
10. 56,810.
12. 56,810,000.

142. Logarithms. Exponents when used in this way for computation are called **logarithms**, abbreviated **log**.

143. The number to which the exponents are applied is called the **base**.

For the purposes of computation the base used is 10.

According to the above definition the equation $30 = 10^{1.48}$ may be written $\log 30 = 1.48$, which is read "The logarithm of 30 is 1.48." These equations mean the same thing; namely, that 1.48 is (approximately) the power of 10 that equals 30.

WRITTEN EXERCISES

Write the following in the notation of logarithms:

1. $700 = 10^{2.84}$.
3. $6 = 10^{0.77}$.
5. $361 = 10^{2.55}$.
2. $75 = 10^{1.87}$.
4. $50 = 10^{1.69}$.
6. $45 = 10^{1.65}$.

Write the following as powers of 10:

7. $\log 20 = 1.3$.
9. $\log 3 = 0.47$.
11. $\log 111 = 2.04$.
8. $\log 500 = 2.70$.
10. $\log 7 = 0.84$.
12. $\log 21 = 1.32$.

144. Characteristic and Mantissa. The integral part of a logarithm is called its **characteristic**, and the decimal part its **mantissa**.

The characteristic of the logarithms of a number greater than unity is one less than the number of digits at the left of the decimal point.

ORAL EXERCISES

1–12. State the characteristic and the mantissa in each of the logarithms in Exercises 1–12 above.

145. Since the characteristics of logarithms can be determined by inspection, tables of logarithms furnish only the mantissas.

EXPLANATION OF THE TABLES

146. The use of the tables, pp. 100 and 101, is best seen from an example.

Find the logarithm of 365.

The first column in the table, p. 100, contains the first two figures of the numbers whose mantissas are given in the table, and the top row contains the third figure.

Hence, find 36 in the left-hand column, p. 100, and 5 at the top.

In the column under 5 and opposite to 36 we find 5623, the required mantissa.

Since 365 is greater than 100 (or 10^2) but less than 1000 (or 10^3), the characteristic of the logarithm is 2.

Therefore, log 365 is 2.5623.

WRITTEN EXERCISES

By use of the table find the logarithms of:

1. 25.
2. 36.
3. 50.
4. 75.
5. 99.
6. 86.
7. 999.
8. 800.
9. 9.9.
10. 8.6.
11. 33,000.
12. 99,900.
13. 1000.
14. 5000.
15. 505.
16. 5.05.

147. Negative Characteristics. An example will serve to show how negative characteristics arise:

From log 346 = 2.5391, we find,

$$\log 34.6 = \log \frac{346}{10} = \log 346 - \log 10 = 2.5391 - 1 = 1.5391.$$

$$\log 3.46 = \log \frac{34.6}{10} = 1.5391 - 1 = 0.5391.$$

$$\log .346 = \log \frac{3.46}{10} = 0.5391 - 1.$$

In the last line we have a positive decimal less 1, and the result is a negative decimal; viz. $-.4609$. But to avoid this change of mantissa, it is customary not to carry out the subtraction, but simply to indicate it. It might be written $-1 + .5391$, but it is customarily abridged into $\bar{1}.5391$. The mantissa is kept positive in all logarithms. The logarithm $\bar{1}.5391$ says that the corresponding number is greater than 10^{-1} (or $\frac{1}{10}$) but less than 10^0 or 1.

We now write $\log .346 = \bar{1}.5391$.

Similarly, $\log .0346 = \log \frac{.346}{10} = \bar{1}.5391 - 1 = \bar{2}.5391$.

Thus we see that the mantissa remains the same, no matter how the position of the decimal point is changed. The mantissa is determined solely by the sequence of digits constituting the number.

The characteristic is determined solely by the position of the decimal point. The characteristics of the logarithm of a number smaller than unity is negative and equals the number of the place occupied by the first significant figure of the decimal.

EXAMPLES

1. What is the characteristic of log .243?

.243 is more than .1 or 10^{-1} but less than 1 or 10^0. Hence,

$$.243 = 10^{-1 + \text{a decimal}}.$$

The characteristic is $\bar{1}$.

2. Similarly, since .00093 is greater than .0001 or 10^{-4} but less than .001 or 10^{-3},

$$\log .00093 = \bar{4} + \text{a decimal}.$$

148. The characteristic having been determined, the mantissa is found from the table in the usual way.

For example:

$$\log .243 = \bar{1}.3856,$$
$$\log .00093 = \bar{4}.9685.$$

WRITTEN EXERCISES

Find the logarithms of:

1. .35.
2. .634.
3. .105.
4. .027.
5. .0023.
6. .0123.
7. .00342.
8. .0004.

149. In finding the number corresponding to a logarithm with negative characteristic, the same method is followed as when the characteristic is positive. The mantissa determines the sequence of digits constituting the number; the characteristic fixes the position of the decimal point.

For example:

If $\log n = \bar{2}.5955$, the digits of n are 394. The characteristic $\bar{2}$ says that n is greater than 10^{-2} (or .01) but less than 10^{-1} (or .1). Hence, the decimal point must be so placed that n has no tenths but some hundredths. Therefore $n = .0394$.

WRITTEN EXERCISES

By use of the tables find the numbers whose logarithms are:

1. $\bar{1}.6232$.
2. $\bar{1}.4914$.
3. 2.4281.
4. 1.9196.
5. 0.9196.
6. 3.4281.
7. $\bar{2}.0792$.
8. $\bar{2}.7076$.
9. 4.9196.
10. 3.2201.
11. 0.2201.
12. 1.2201.
13. $\bar{3}.0969$.
14. $\bar{1}.6972$.
15. 3.9284.
16. 2.9284.
17. 1.9284.
18. 5.7832.

Find n if:

19. $\log n = 1.9289$.
20. $\log n = 0.9289$.
21. $\log (n - 1) = 3.9294$.
22. $\log (\frac{1}{2} n) = 1.6128$.

USE OF THE TABLES FOR COMPUTATION

150. For use in computation by logarithms the laws of exponents may be expressed thus:

I. $10^m \cdot 10^r = 10^{m+r}$. *The logarithm of a product is the sum of the logarithms of the factors.*

II. $\frac{10^m}{10^r} = 10^{m-r}$. *The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

III. $(10^m)^r = 10^{mr}$. *The logarithm of a number with an exponent is the product of the exponent and the logarithm of the number.*

Since r may be positive or negative, integral or fractional, Law III provides not only for raising to integral powers, but also for finding reciprocals of such powers, and for extracting roots.

EXAMPLES

1. Multiply 21 by 37.

   1. log 21 = 1.3222, table, p. 100.
   2. log 37 = 1.5682, table, p. 100.
   3. Adding, log 21 + log 37, or log $(21 \times 37) = 2.8904$ (Sec. 150).
   4. $\therefore 21 \times 37 = 777$ from table, p. 101.

2. Divide 814 by 37.

   1. log 814 = 2.9106, table, p. 101.
   2. log 37 = 1.5682, table, p. 100.
   3. $\therefore$ log 814 − log 37 = 2.9106 − 1.5682 = 1.3424.
   4. $\therefore 814 \div 37 = 22$, table, p. 100.

3. Extract the cube root of 729.

   1. $\sqrt[3]{729} = (729)^{\frac{1}{3}}$.
   2. $\log 729^{\frac{1}{3}} = \frac{1}{3} \log 729$ (Sec. 150).
   3. log 729 = 2.8627, table, p. 101.
   4. $\frac{1}{3}$ of 2.8627 = 0.9542.
   5. $\therefore (729)^{\frac{1}{3}}$ or $\sqrt[3]{729} = 9$, table, p. 101.

WRITTEN EXERCISES

Compute by use of logarithms:

1. 8×15.
2. 41×23.
3. 37×17.
4. 12×17.
5. $893 \div 19$.
6. 5^4.
7. 31^2.
8. $\sqrt{484}$.
9. $\sqrt[3]{343}$.
10. $940 \div 47$.
11. 19^2.
12. 7^3.
13. 4^4.
14. $\sqrt{196}$.
15. $\sqrt[3]{216}$.

Since the logarithm is approximate, the result in general is approximate. Thus, log $\sqrt[4]{(256)} = 0.60205$, which is not the logarithm of 4, but is sufficiently near to be recognized in the table.

151. How to calculate the logarithm of a number not found in the table is best seen from an example.

EXAMPLE

Find the logarithm of 257.3 (see 16th line of table, p. 100):

	0	1	2	3	4	5	6	7	8	9
25	3979	3997	4014	4031	4048	4065	4082	**4099**	**4116**	4133

The numbers nearest to 257.3, whose logarithms are given in the table, are 257 and 258. We have

$$\begin{aligned} \log 258 &= 2.\mathbf{4116} \\ \log 257 &= 2.\mathbf{4099} \\ \hline \text{Difference,}\quad &.0017 \end{aligned}$$

That is, an increase of 1 in the number causes an increase of .0017 in the logarithm. Assuming that an increase of .3 in the number would cause an increase of .3 of .0017, or .0005, in the logarithm, then

$$\log 257.3 = 2.4099 + .0005 = 2.4104.$$

NOTES: 1. The product of .3 and .0017 is .00051, but we take only four places, because the mantissas as given in the table are expressed to four places only. If the digit in the fifth decimal place of the correction is more than 5, we replace it by a unit in the fourth place.

2. The difference between two succeeding mantissas of the table (called the **tabular difference**) can be seen by inspection.

3. What is written in finding the logarithm of 257.3 should be at most the following:

$$\begin{aligned} \log 257 &= 2.4099 \\ \text{correction for } .3 &= 5 \\ \hline \log 257.3 &= 2.4104 \end{aligned} \qquad\qquad \begin{aligned} \text{tabular difference}\ &17 \\ &\underline{.3} \\ &5.1 \end{aligned}$$

4. The corrections are made on the assumption that the change in the logarithm is proportional to the change in the number. This is sufficiently accurate when used within the narrow limits here prescribed.

WRITTEN EXERCISES

Find the logarithm of:

1. 1235.
2. 23.5.
3. 2.36.
4. .0237.
5. 1425.
6. 1837.
7. 6720.
8. 67.25.
9. 3.142.
10. 1.414.
11. 1.732.
12. .6226.
13. .4071.
14. 85.51.
15. .0125.
16. .4267.
17. .3002.
18. 9009.
19. 12.02.
20. 5.008.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

152. To calculate the number whose logarithm is given apply the table as follows:

(1) If the given logarithm is in the table, the number can be seen at once.

(2) If the given logarithm is not in the table, the number corresponding to the nearest logarithm of the table may be taken.

A somewhat closer approximation may be found by using the method of the following example:

EXAMPLE

Find the number whose logarithm is 1.4271. The mantissas nearest to this are found in the 17th line of table, p. 100.

	0	1	2	3	4	5	6	7	8	9
26	4150	4166	4183	4200	4216	4232	4249	**4265**	**4281**	4298

The process is the reverse of that of finding the logarithm.

The next smaller mantissa in the table is 4265, corresponding to the number 267. The difference between this mantissa and the given mantissa is 6. The tabular difference between 4265 and the next larger mantissa is 16. An increase of 16 in the logarithm corresponds to an increase of 1 in the number. Hence, an increase of 6 in the logarithm corresponds to an increase of $\frac{6}{16}$ of 1, or .4, in the number. This means .4 of one unit in the number 267. What its place value is in the final result depends upon the characteristic. The digits of the result are 2674.

The characteristic 1 shows that the desired number is greater than the first power of 10, but less than the second power of 10 or 100. Hence, the decimal point must be placed between 6 and 7, and the final result is 26.74.

NOTES: 1. For reasons similar to those of Note 1, p. 99, the correction should be carried to one place only.

2. At most the following should be written:

$$\begin{array}{rrl} \log n = & 1.4271 & \qquad \text{tab. diff. } 16 \\ \underline{\text{mantissa for } 267 =} & \underline{.4265} & \qquad \tfrac{6}{16} = .4 \\ \text{diff.} & 6 & \end{array}$$

Therefore, $n = 26.74.$

WRITTEN EXERCISES

Find the number whose logarithm is:

1. 0.7305.
2. 0.5029.
3. 1.4682.
4. 2.9023.
5. 3.1467.
6. 3.6020.
7. $\bar{1}.1962$.
8. $\bar{2}.0342$.
9. $\bar{3}.3920$.
10. $\bar{3}.9485$.
11. 4.6987.
12. $\bar{2}.6376$.

153. Exponential Equations. When the unknown quantity occurs in an equation as an exponent, the equation is called an **exponential equation.**

EXAMPLE

1. Solve: $3^x = 729.$ *(1)*

Taking the log of both members, $\log 3^x = \log 729.$ *(2)*

Or, $x \cdot \log 3 = \log 729.$ Sec. 150, III. *(3)*

Dividing by log 3, $x = \dfrac{\log 729}{\log 3} = \dfrac{2.8627}{.4771} = 6.$ *(4)*

Test: $3^6 = 729.$

The test is especially important because the division in step (*4*) is not exact. The approximation is so close, however, that the correct number will always be suggested, and the test will finally verify it.

It should also be noted that step (*4*) contains the quotient of two logarithms, which is not the same as the logarithm of the quotient; hence, Sec. 150, II, is not applicable.

2. Solve: $8^{7-2y} = 512.$ *(1)*

$(7 - 2y)\log 8 = \log 512.$ *(2)*

$\therefore 7 - 2y = \dfrac{\log 512}{\log 8} = \dfrac{2.7093}{.9031} = 3.$ *(3)*

$\therefore -2y = -4, \text{ or } y = 2.$ *(4)*

Test: $8^{7-2\cdot 2} = 8^3 = 512.$

WRITTEN EXERCISES

Solve:

1. $2^x = 512.$
2. $3^y = 243.$
3. $2^{2x} = 256.$
4. $8^w = 4096.$
5. $10^{z-2} = 1000.$
6. $10^y = 500.$

7. $12^x = 20,736.$ 10. $15^{a-x} = 3375.$

8. $9^{6-x} = 243.$ 11. $a^x = b^{x-1}.$

9. $7^{2x} = 343.$ 12. $c^{x+1} = e^{-x}.$

13. Solve for x and y, computing the values to 2 decimal places:

$$\begin{cases} 3^x = 2^y. \\ 3^{x-2} = 8^y. \end{cases}$$

REVIEW

WRITTEN EXERCISES

Find exactly or approximately by use of logarithms the value of:

1. $\sqrt{2}.$ 4. $\sqrt[5]{7}.$ 7. $\sqrt[3]{756}.$ 10. $(1.03)^7.$

2. $\sqrt{5}.$ 5. $\sqrt{926}.$ 8. $\sqrt[5]{812}.$ 11. $(1.04)^{10}.$

3. $\sqrt[3]{9}.$ 6. $\sqrt{656}.$ 9. $(1.5)^4.$ 12. $(1.06)^9.$

13. $\dfrac{(164)(798)}{779}.$ 15. $\sqrt{(624)(598)(178)}.$

14. $\dfrac{(732)(774)}{(731)(671)}.$ 16. $\sqrt{\dfrac{(651)(654)(558)}{763}}.$

17. It is known that the volume of a sphere is $\frac{4}{3}\pi r^3$, r being the length of the radius. Using 3.14 as the approximate value of π, find by logarithms the volume of a sphere of radius 7.3 in.

18. Find, as above, the volume of a sphere whose radius is 36.4 ft.

Calculate by logarithms:

19. $\dfrac{(132)(1837)}{167}.$ 22. $\sqrt{\dfrac{(294)(1842)}{307}}.$

20. $\dfrac{(2076)(379)}{173}.$ 23. $\sqrt{\dfrac{(2373)(675)}{113}}.$

21. $\dfrac{(3059)(349)}{(19)(23)(2443)}.$ 24. $\sqrt[3]{.0578}.$

25. $\dfrac{(15.61)^2}{(700)^{\frac{1}{3}}}$.

26. $\sqrt{\dfrac{(7688)(7719)}{(248)(249)}}$.

27. $(217.6)(.00681)$.

28. $\dfrac{[\sqrt{278.2}(2.578)]^2}{\sqrt[3]{.00231}\cdot\sqrt{76.19}}$.

29. Given $a = 0.4916$, $c = 0.7544$, and $b = c^2 - a^2$. Find b.

Suggestion. $b = (c - a)(c + a)$.

30. It is known that in steam engines, the piston head's average velocity (c) per second is approximately given by the formula:

$$c = 1.7\sqrt[3]{s\sqrt{\frac{p}{15}}},$$

where s denotes the distance over which the piston moves (expressed in the same unit as c), and p the number of pounds pressure in the cylinder.

(1) Find c, if $s = 32.5$ in., $p = 110$ lb.

(2) Find p, if $c = 15$ ft., $s = 2.6$ ft.

31. Solve for x and y, computing decimal results to 2 decimal places:

$$2^x - 6^y = 0,$$
$$2^{x+1} - 12^y = 0.$$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What is a *logarithm?* Sec. 142.

2. What is the meaning of *base* in logarithms? Sec. 143.

3. What is the integral part of a logarithm called? What is the decimal part called? Sec. 144.

4. State how to find the logarithm of a product. Sec. 150, I.

5. State how to find the logarithm of a quotient. Sec. 150, II.

6. How is the logarithm of a power found? Sec 150, III.

7. State each of the processes named in statements 4, 5, and 6 in the form of laws of exponents. Sec. 150.

HISTORICAL NOTE

It is remarkable that the discovery of Logarithms occurred so timely, for this great machine of computation was invented by Napier at the beginning of the seventeenth century, just in time to aid the new work in astronomy and navigation; Galileo had devised the telescope and Kepler was ready to calculate the orbits of the planets. It is also remarkable that Napier worked out the principle of Logarithms without the use of exponents, and this peculiar method, which is too complex to be explained here, produced tables quite different from those now in use. As soon as Napier's great work on Logarithms was published, Henry Briggs, a teacher in Gresham College, London, hastened to visit Napier, and suggested the advantages of the base 10, and thus laid the foundation of our tables of common Logarithms.

John Napier, also known as the Baron of Merchiston, was a Scotchman, born in 1550, and published his work on Logarithms in 1614, only three years before his death. The modesty and simplicity of the great Scottish philosopher is shown by his attitude toward Briggs, for when informed that the latter had been obliged to postpone his promised visit, Napier regretfully replied, "Ah, Mr. Briggs will not come."

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Napier's discovery, whose importance is not exaggerated by the claim that "it doubled the life of the astronomer by shortening his labor," followed immediately upon the general acceptance of the Hindoo notation and the introduction by Stevin of decimal fractions. Thus, the seventeenth century saw the perfection of the three greatest instruments of modern calculation, the Hindoo notation, decimal fractions, and Logarithms.

CHAPTER V

IMAGINARY AND COMPLEX NUMBERS

154. Imaginary Numbers. The numbers defined in what precedes have all had positive squares. Consequently, among them the equation $x^2 = -3$, which asks, "What is the number whose square is -3?" has no solution.

A solution is provided by defining a new number, $\sqrt{-3}$, as a number whose square is -3. Similarly we define $\sqrt{-a}$, a being a positive number, as a number whose square is $-a$.

The square roots of negative numbers are called **imaginary numbers.**

155. If a is positive, $\sqrt{-a}$ may be expressed $\sqrt{a}\sqrt{-1}$.

Similarly, $\sqrt{-49} = \sqrt{49(-1)} = 7\sqrt{-1}$.

156. Real Numbers. In distinction from imaginary numbers, the numbers hitherto studied are called **real numbers.**

WRITTEN EXERCISES

Express as in Section 155:

1. $\sqrt{-9}$.
2. $\sqrt{-16}$.
3. $\sqrt{-25}$.
4. $\sqrt{-100}$.
5. $-\sqrt{-64}$.
6. $\sqrt{-8}$.
7. $\sqrt{-18}$.
8. $-\sqrt{-32}$.
9. $-\sqrt{-7}$.
10. $\sqrt{-12}$.
11. $\sqrt{-50}$.
12. $-\sqrt{-75}$.

157. The positive square root of -1 is frequently denoted by the symbol i; that is, $\sqrt{-1} = i$.

Using this we write:

$$\sqrt{-5} = \sqrt{5} \cdot i; \quad \sqrt{-49} = \pm 7i; \quad \text{also } \sqrt{-75a^2b} = 5a\sqrt{3b} \cdot i.$$

NOTE. Throughout this chapter the radical sign is taken as positive.

WRITTEN EXERCISES

Rewrite the following, using the symbol i as in Section 157:

1. $2+\sqrt{-4}$.
2. $3-\sqrt{-9}$.
3. $4+\sqrt{-4}$.
4. $5-\sqrt{-16}$.
5. $25-\sqrt{-25}$.
6. $5-\sqrt{-3}$.
7. $3+\sqrt{-6}$.
8. $7+\sqrt{-12}$.
9. $12-\sqrt{-9}$.
10. $2\sqrt{-100}$.
11. $4\sqrt{-(a+b)}$.
12. $\sqrt{a}+\sqrt{-b^2c^2}$.
13. $-\sqrt{-b^2c}$.
14. $a+\sqrt{-(a^2+x^2)}$.
15. $x+y-\sqrt{-xy^2}$.
16. $p^2+\sqrt{-(p+q)^3}$.
17. $\sqrt{x}-\sqrt{-(a+b)^2}$.
18. $\sqrt{x+y}+\sqrt{-(x+y)^4}$.

158. Complex Numbers. A binomial one of whose terms is real and the other imaginary is called a **complex number.**

The general form of a complex number is $a+bi$, where a and b may be any real numbers.

NOTE. Complex numbers are also simply called **imaginary,** any expression which involves i being called imaginary. Single terms in which i is a factor (those which we have called imaginary above) are often called **pure imaginaries,** while the others are called **complex imaginaries.** Thus, $\sqrt{-2}$, $3\sqrt{-a}$, $5i$ are pure imaginaries and $1-\sqrt{-3}$, $a-\sqrt{-b}$ are complex imaginaries.

159. Processes with Imaginary and Complex Numbers. After introducing the symbol i for the imaginary unit $\sqrt{-1}$, the operations with imaginary and complex numbers are performed like the operations with real numbers.

I. *Addition and Subtraction.*

EXAMPLE

Add $\sqrt{-9}$, $-\sqrt{-25}$, $\sqrt{-3}$.

$$\begin{aligned}\sqrt{-9} &= 3i.\\ -\sqrt{-25} &= -5i.\\ \sqrt{-3} &= \sqrt{3}\cdot i.\end{aligned}$$

$\therefore$ the sum is $(3-5+\sqrt{3})i = -(2-\sqrt{3})i$.

162. By means of the values of i^2, i^3, i^4, any power of i can be shown to be either $\pm i$ or ± 1.

For example: $i^{63} = i^{60} \cdot i^3 = (i^4)^{15} \cdot i^3 = 1^{15} \cdot i^3 = i^3 = -i.$

WRITTEN EXERCISES

Simplify similarly:

1. i^9.
2. i^{10}.
3. i^{12}.
4. i^{16}.
5. i^{21}.
6. i^{27}.
7. i^{54}.
8. i^{56}.
9. i^{198}.
10. i^{143}.
11. i^{400}.
12. i^{3001}.

Perform the operations indicated:

13. $(1+i)^2$.
14. $(1-i)^3$.
15. $(1-i)^3 \cdot i^4$.
16. $\left(\frac{-1+3i}{2}\right)^3$.
17. $(1+i) \cdot i^6$.
18. $\left(\frac{-1-3i}{2}\right)^3$.
19. $(1+i) \cdot (1-i)^2$.
20. $(1+i)^2 \div (1-i)^2$.

IMAGINARIES AS ROOTS OF EQUATIONS

163. Complex numbers often occur as roots of quadratic equations.

EXAMPLE

Solve: $$x^2 + x + 1 = 0. \qquad (1)$$

$$x^2 + x = -1. \qquad (2)$$

Completing the square, $$x^2 + x + \tfrac{1}{4} = \tfrac{1}{4} - 1. \qquad (3)$$

$$\therefore x + \tfrac{1}{2} = \pm\sqrt{-\tfrac{3}{4}}. \qquad (4)$$

$$\therefore x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i. \qquad (5)$$

TEST: $(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i)^2 + (-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i) + 1 = 0.$

WRITTEN EXERCISES

Solve and test, expressing the imaginary roots in the form $a + bi$:

1. $x^2 + 5 = 0$.
2. $x^2 + 2x + 2 = 0$.
3. $x^2 + 2x + 37 = 0$.
4. $x^2 - 8x + 25 = 0$.
5. $x^2 - 6x + 10 = 0$.
6. $m^2 + 4m + 85 = 0$.
7. $x^2 + 10x + 41 = 0$.
8. $x^2 + 30x + 234 = 0$.

7. $4+i$ by $5-i$.
8. $a+xi$ by $a-xi$.
9. a^2+b^2i by a^2-b^2i.
10. $\sqrt{r}+3i$ by $\sqrt{r}-3i$.
11. $\sqrt{-25}$ by $\sqrt{-9}$ by $\sqrt{-5}$.
12. $\sqrt{-a}$ by $\sqrt{-b}$ by $-ci$.

III. *Division.*

Fractions (that is, indicated quotients) may be simplified by rationalizing the denominator (Sec. 364, p. 264, Part I).

For example:

1. $\frac{\sqrt{-7}}{\sqrt{-5}}=\frac{\sqrt{-7}\sqrt{-5}}{\sqrt{-5}\sqrt{-5}}=\frac{\sqrt{7}\cdot\sqrt{5}(-1)}{-5}=\frac{\sqrt{35}}{5}$.

2. $\frac{2+\sqrt{-3}}{3-\sqrt{-5}}=\frac{(2+\sqrt{-3})(3+\sqrt{-5})}{(3-\sqrt{-5})(3+\sqrt{-5})}=\frac{6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15}}{9-(-5)}$
$=\frac{1}{14}(6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15})$.

3. $\frac{x+yi}{x-yi}=\frac{(x+yi)^2}{(x-yi)(x+yi)}=\frac{x^2+2xyi-y^2}{x^2+y^2}$.

WRITTEN EXERCISES

Write in fractional form and rationalize the denominators:

1. $\sqrt{-6}\div\sqrt{2}$.
2. $1\div(a+xi)$.
3. $\sqrt{-3}\div\sqrt{-5}$.
4. $\sqrt{ax}\div\sqrt{-a}$.
5. $1\div(2-\sqrt{-3})$.
6. $4\sqrt{-1}\div-2\sqrt{-4}$.
7. $a\div(a-bi)$.
8. $(a+bi)\div(a-bi)$.
9. $(3+6i)\div(5+4i)$.
10. $(\sqrt{3}-9i)\div(\sqrt{2}-9i)$.
11. $(x-\sqrt{-7})\div(x+\sqrt{-7})$.
12. $\left(a-\frac{\sqrt{-5}}{2}\right)\div\left(a+\frac{\sqrt{-5}}{2}\right)$.
13. $(\sqrt{-2}+\sqrt{-5})\div(\sqrt{-5}-\sqrt{-2})$.

161. Powers of the Imaginary Unit. Beginning with $i^2=-1$ and multiplying successively by i we find:

$i^2=-1$.

$i^3=i^2\cdot i=-i$.

$i^4=i^2\cdot i^2=-1(-1)=+1$.

$i^5=i^4\cdot i=i$.

$i^6=i^4\cdot i^2=i^2=-1$.

$i^7=i^6\cdot i=-1\cdot i=-i$.

$i^8=i^4\cdot i^4=(+1)^2=+1$.

$i^9=i^8\cdot i=i$.

162. By means of the values of i^2, i^3, i^4, any power of i can be shown to be either $\pm i$ or ± 1.

For example: $i^{63} = i^{60} \cdot i^3 = (i^4)^{15} \cdot i^3 = 1^{15} \cdot i^3 = i^3 = -i.$

WRITTEN EXERCISES

Simplify similarly:

1. i^9.
2. i^{10}.
3. i^{12}.
4. i^{16}.
5. i^{21}.
6. i^{27}.
7. i^{54}.
8. i^{56}.
9. i^{198}.
10. i^{143}.
11. i^{400}.
12. i^{3001}.

Perform the operations indicated:

13. $(1+i)^2$.
14. $(1-i)^3$.
15. $(1-i)^3 \cdot i^4$.
16. $\left(\dfrac{-1+3i}{2}\right)^3$.
17. $(1+i) \cdot i^6$.
18. $\left(\dfrac{-1-3i}{2}\right)^3$.
19. $(1+i) \cdot (1-i)^2$.
20. $(1+i)^2 \div (1-i)^2$.

IMAGINARIES AS ROOTS OF EQUATIONS

163. Complex numbers often occur as roots of quadratic equations.

EXAMPLE

Solve:
$$x^2 + x + 1 = 0. \qquad (1)$$
$$x^2 + x = -1. \qquad (2)$$
Completing the square, $x^2 + x + \frac{1}{4} = \frac{1}{4} - 1. \qquad (3)$
$$\therefore x + \tfrac{1}{2} = \pm\sqrt{-\tfrac{3}{4}}. \qquad (4)$$
$$\therefore x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i. \qquad (5)$$

Test: $(-\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \cdot i)^2 + (-\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \cdot i) + 1 = 0.$

WRITTEN EXERCISES

Solve and test, expressing the imaginary roots in the form $a + bi$:

1. $x^2 + 5 = 0$.
2. $x^2 + 2x + 2 = 0$.
3. $x^2 + 2x + 37 = 0$.
4. $x^2 - 8x + 25 = 0$.
5. $x^2 - 6x + 10 = 0$.
6. $m^2 + 4m + 85 = 0$.
7. $x^2 + 10x + 41 = 0$.
8. $x^2 + 30x + 234 = 0$.

9. $y^2 - 4y + 53 = 0.$
10. $z^2 - 6z + 90 = 0.$
11. $p^2 + 20p + 104 = 0.$
12. $2x^2 + 4x + 3 = 0.$
13. $3x^2 + 2x + 1 = 0.$
14. $12t^2 + 24 = 0.$
15. $6w^2 + 30 = 0.$
16. $x^2 - x + 1 = 0.$
17. $4x^2 + 4x + 3 = 0.$
18. $x^2 + x + 5 = 0.$
19. $6x^2 + 3x + 1 = 0.$
20. $10x^2 - 2x + 3 = 0.$
21. $7t^2 - t + 1 = 0.$
22. $12x^2 + x + 1 = 0.$
23. $8t^2 + t + 6 = 0.$
24. $7x^2 + x + 5 = 0.$
25. $15z^2 + 5z - 1 = 0.$
26. $8v^2 + 3v + 6 = 0.$

164. The occurrence of imaginary roots in solving equations derived from problems often indicates the impossibility of the given conditions.

EXAMPLE

A rectangular room is twice as long as it is wide; if its length is increased by 20 ft. and its width diminished by 2 ft., its area is doubled. Find its dimensions.

Solution. 1. Let x = the width of the room, and $2x$ its length.
2. Then $(2x + 20)(x - 2) = 2 \cdot 2x \cdot x$, or $x^2 - 8x + 20 = 0.$
3. Solving (2), $x = 4 \pm 2i.$

The fact that the results are complex numbers shows that no actual room can satisfy the conditions of the problem.

WRITTEN EXERCISES

Solve and determine if the problems are possible:

1. In remodeling a house a room 16 ft. square is lengthened on one side a certain number of feet and decreased on the other by twice that number. If the area of the room as changed is 296 sq. ft., what are the original dimensions?

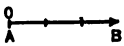
2. A triangle has an altitude 2 in. greater than its base, if it has an area of 32 sq. ft., find the length of its base.

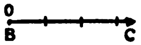
3. If a train moving x mi. per hour travels 90 mi. in $15 - x$ hours, what is its rate per hour?

GRAPHICAL REPRESENTATION

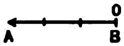
165. We have seen that positive integers and fractions can be represented by lines.

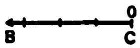
Thus, the line AB represents 3, and the line BC represents $3\frac{1}{2}$.





Similarly, we have seen that negative integers and fractions, which for a long time were considered to be meaningless, can be represented by lines.

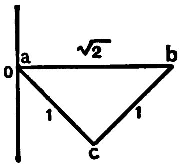


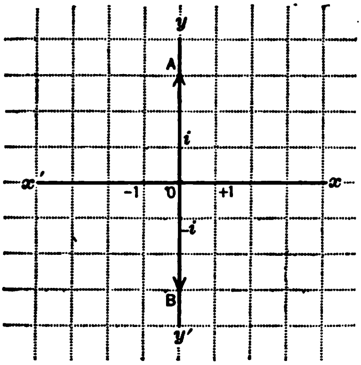


Thus, the line BA represents -3, and the line CB represents $-3\frac{1}{2}$.

Irrational numbers can also be represented by lines.

Thus, in the right-angled triangle abc, the line ab represents the $\sqrt{2}$.





Like the negative number the imaginary number remained uninterpreted several centuries. But this number also can be represented graphically.

Thus, if a unit length on the y-axis be chosen to represent $\sqrt{-1}$ or i, the negative unit $-\sqrt{-1}$ or $-i$ should evidently be laid off in the opposite direction. $3\sqrt{-1}$ or $3i$ would then be represented by OA and $-3i$ by OB, as in the figure, and others similarly.

The reason for placing $\sqrt{-1}$ or i on a line at right angles to the line on which real numbers are plotted may be seen in the fact that multiplying 1 by $\sqrt{-1}$ twice changes $+1$ into -1. On the graph $+1$ can be changed into -1 by turning it through 180°. If multiplying 1 by $\sqrt{-1}$ twice turns the line 1 through 180°, multiplying 1 by $\sqrt{-1}$ once should turn $+1$ through 90°.

For example:

1. Represent graphically $\sqrt{-4}$:

$\sqrt{-4}=\sqrt{4}\,i=2\,i$; this is represented by a line 2 spaces long drawn upward on the y-axis.

2. Represent graphically $-\sqrt{-3}$:

$-\sqrt{-3}=-\sqrt{3}\,i=-1.7\,i$ (approximately); this is represented by a line 1.7 spaces long drawn downward on the y-axis.

WRITTEN EXERCISES

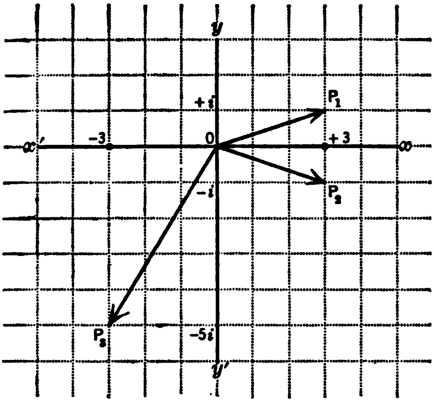
Represent graphically:

1. $3\,i$.
2. $-2\,i$.
3. $\sqrt{-9}$.
4. $\sqrt{-16}$.
5. $-5\,i$.
6. $5\,i$.
7. $\sqrt{-3}$.
8. $-\sqrt{-12}$.
9. $-5\sqrt{-4}$.
10. $-3\,i$.
11. $+2\sqrt{-3}$.
12. $5\sqrt{-9}$.

Complex numbers may be represented graphically by a modification of the plan used in representing imaginary numbers.

EXAMPLES

1. Represent graphically $3+i$.



To do this 3 is laid off on the axis of real numbers (xx'), and i upward on the axis of imaginaries (yy'). As in other graphical work this locates the point P_1 which is taken to represent the complex number, $3+i$.

The number $\sqrt{a^2+b^2}$ is called the *modulus* of the complex number $a+bi$. As appears from the figure $OP_1=\sqrt{3^2+1^2}$, and hence OP_1 represents the modulus of $3+i$.

2. Represent graphically $3 - i$.

The point P_2 is the graph of the complex number $3 - i$, and OP_2 represents its modulus.

3. Represent graphically $-3 - 5i$.

The point P_3 is the graph of the complex number $-3 - 5i$, and OP_3 represents its modulus.

We have thus *interpreted by means of diagrams* positive and negative integers, positive and negative fractions, positive and negative irrational numbers, and positive and negative complex numbers; in fact, all of the numbers used in elementary algebra.

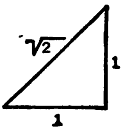
SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What is an *imaginary number?* Sec. 154.
2. What term is used to designate numbers not imaginary? Sec. 156.
3. Define and illustrate a *complex number.* Sec. 158.
4. What is the meaning of the symbol i in complex numbers? Sec. 158.
5. How may any integral power of i be expressed? Secs. 161, 162.
6. How do imaginary numbers often occur in practice? What do imaginary roots indicate in solving problems? Secs. 163, 164.

HISTORICAL NOTE

Negative numbers were so long a stumbling block to mathematicians that their square roots were naturally regarded as impossible until recent times. The Greeks understood irrationals and could express many of these numbers concretely; for example, the $\sqrt{2}$ was shown by them to be the length of the hypotenuse of a right triangle whose sides are unity, as in the figure, but the $\sqrt{-2}$ had no meaning to them. The Hindoo, Bhaskara, said:

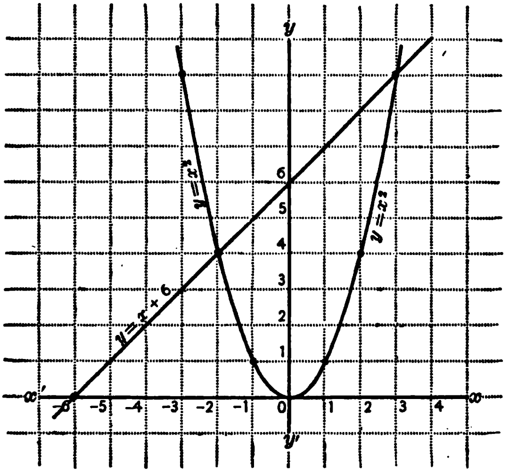


"The square of a positive, as also of a negative number, is positive, but there is no square root of a negative number, for it is not a square." Even the great scholars of the sixteenth and seventeenth centuries did little more than to accept imaginaries as numbers, and it remained for Caspar Wessel (1797) to make the first concrete representation of complex numbers; but his discovery made little impression until Gauss emphasized its importance.

Karl Friedrich Gauss was born at Brunswick, Germany, in 1777. His father was a mason and took little interest in his son's education, but the boy's wonderful genius for numbers attracted the attention of his teachers, who induced the Duke of Brunswick to send young Gauss to a preparatory school. He entered the University of Göttingen in 1795 and soon made discoveries in the properties of numbers that won for him high rank among mathematicians. On the appearance of his great work, *Disquisitiones Arithmeticæ*, published in 1801 when Gauss was only twenty-four years old, his contemporary, Laplace, declared Gauss to be the greatest mathematician of all Europe. Gauss died in 1855 after a life devoted to mathematics. He enriched all its branches, including its applications in astronomy and physics, with the lasting products of his wonderful genius. He has been worthily called: *Princeps Mathematicorum*, the Prince of Mathematicians.

KARL FRIEDRICH GAUSS

Notes: 1. Step 2 may be done by simply noting two points of the graph of $y = x + 6$ and laying a ruler connecting them. The roots can be read while the ruler is in position, and thus the same graph for $y = x^2$ can be used for several solutions.



2. The equation must first be put in the form $x^2 + px + q = 0$, if not so given.

WRITTEN EXERCISES

Solve graphically:

1. $x^2 - 5x + 6 = 0.$
2. $x^2 + 3x + 2 = 0.$
3. $x^2 - 2x - 3 = 0.$
4. $x^2 + 2x - 3 = 0.$
5. $x^2 - 3x - 40 = 0.$
6. $x^2 + 4x + 4 = 0.$
7. $2x^2 - x - 1 = 0.$
8. $3x^2 - 2x - 1 = 0.$
9. $x^2 + x + \frac{1}{4} = 0.$
10. $x^2 + x - 2 = 0.$
11. $4x^2 + 4x + 1 = 0.$
12. $x^2 - 9 = 0.$
13. $x^2 - 4x + 4 = 0.$
14. $x^2 + \frac{2}{3}x + \frac{1}{9} = 0.$
15. $x^2 - 4x + 3 = 0.$
16. $x^2 - 4x - 5 = 0.$
17. $x^2 - 7x + 12 = 0.$
18. $x^2 - 3x - 10 = 0.$

WRITTEN EXERCISES

1. Construct on a large sheet of squared paper the points corresponding to the table of squares given below.

2. Then sketch a smooth curve through the points beginning with $-5, 25$.

Number	Square
-5	25
-4	16
-3	9
-2	4
-1	1
-0	0
1	1
2	4
3	9
4	16
5	25

The work should be carefully done, and the result preserved for later use. As there are no negative values of x^2, the x-axis should be taken near the lower edge of the paper. The unit chosen should be quite large; for example, 10 spaces. Then the table might include squares of numbers increasing by tenths: 1, 1.1, 1.3, etc. The curve will be a graphical table of squares and square roots.

3. Read to one decimal place from the graph $\sqrt{2}$; $\sqrt{3}$; $\sqrt{5}$; $\sqrt{6}$; $\sqrt{7}$; $\sqrt{8}$.

Every y-distance is the square of the corresponding x-distance; and every x-distance is the square root of the corresponding y-distance. We see that for every y-distance there are two corresponding x-distances, one plus and the other minus, corresponding to the two square roots. Thus the points of the curve for which $y = 4$ are those whose values of x are 2 and -2, respectively, *i.e.* $\sqrt{4} = \pm 2$.

168. Graphical Solution of Quadratic Equations. Any value of x which satisfies the system

$$\begin{cases} y = x^2, \\ y = -px - q \end{cases}$$

makes x^2 equal to $-px - q$, or $x^2 + px + q = 0$.

The values of x satisfying the system may be read from the graph of $y = x^2$.

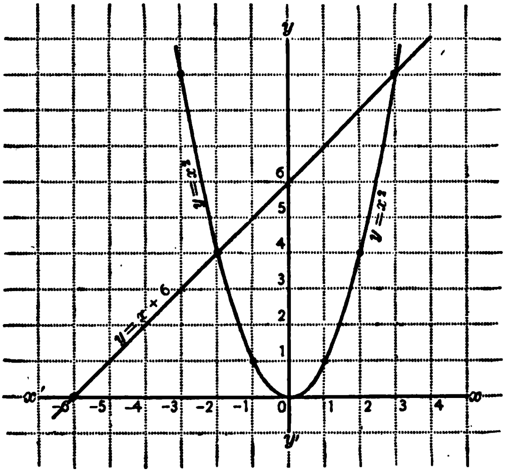
EXAMPLE

Solve graphically $x^2 - x - 6 = 0$.

1. Construct the graph of $y = x^2$. (As in Sec. 167.)
2. Construct the graph of $y = x + 6$.
3. They intersect at points for which $x = -2$ and $+3$.

$\therefore$ the roots of $x^2 - x - 6 = 0$ are $-2, 3$.

Notes: 1. Step 2 may be done by simply noting two points of the graph of $y = x + 6$ and laying a ruler connecting them. The roots can be read while the ruler is in position, and thus the same graph for $y = x^2$ can be used for several solutions.



2. The equation must first be put in the form $x^2 + px + q = 0$, if not so given.

WRITTEN EXERCISES

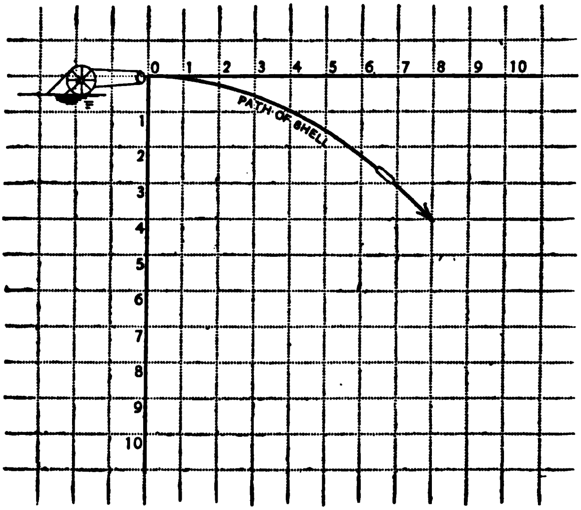
Solve graphically:

1. $x^2 - 5x + 6 = 0$.
2. $x^2 + 3x + 2 = 0$.
3. $x^2 - 2x - 3 = 0$.
4. $x^2 + 2x - 3 = 0$.
5. $x^2 - 3x - 40 = 0$.
6. $x^2 + 4x + 4 = 0$.
7. $2x^2 - x - 1 = 0$.
8. $3x^2 - 2x - 1 = 0$.
9. $x^2 + x + \frac{1}{4} = 0$.
10. $x^2 + x - 2 = 0$.
11. $4x^2 + 4x + 1 = 0$.
12. $x^2 - 9 = 0$.
13. $x^2 - 4x + 4 = 0$.
14. $x^2 + \frac{2}{3}x + \frac{1}{9} = 0$.
15. $x^2 - 4x + 3 = 0$.
16. $x^2 - 4x - 5 = 0$.
17. $x^2 - 7x + 12 = 0$.
18. $x^2 - 3x - 10 = 0$.

19. The path of a projectile fired horizontally with a given velocity from an elevation, as at 0 in the figure, may be represented by the graph of the equation $y = \frac{gx^2}{2v^2}$, where $g = 32$ and v is the initial velocity of the projectile in feet per second. Let $v = 16$ ft. per second and compute the numbers to complete the table of values of x and y.

TABLE

x	y
0	0
1	( )
4	( )
8	( )
9	( )
16	( )
24	( )
32	( )
48	( )



Read from the graph of this table the horizontal distance traveled by the projectile when it is 4 ft. below the starting point.

20. Construct similarly the path of a projectile whose initial velocity is 32 ft. per second.

21. A cannon of a fort on a hill is 300 ft. above the plane of its base. The cannon can be charged so as to give the projectile an initial velocity of 100 ft. per second. How far does a projectile travel horizontally before it strikes the plane?

22. The enemy is observed at a point $2\frac{1}{2}$ mi. from the foot of the vertical line in which the cannon stands. With what initial velocity must the ball start to strike the enemy?

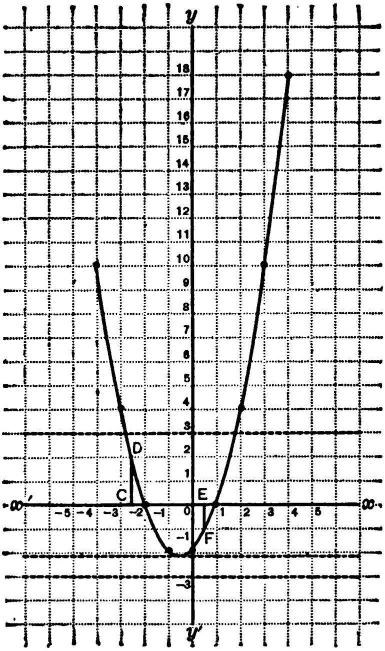
169. Graphs of Quadratic Functions. The solutions of $ax^2 + bx + c = 0$ may also be represented graphically by plotting the curve corresponding to $y = ax^2 + bx + c$. The real roots of the equation will be represented by the points where the curve cuts the x-axis.

EXAMPLE

Represent the solutions of $x^2 + x - 2 = 0$.

To draw the curve $y = x^2 + x - 2$, we make a table of a sufficient number of pairs of values of x and y, plot the corresponding points, and sketch a smooth curve through them.

x	y
-4	10
-3	4
-2	0
-1	-2
$-\frac{3}{4}$	$-2\frac{3}{16}$
$-\frac{1}{2}$	$-2\frac{1}{4}$
$-\frac{1}{4}$	$-2\frac{3}{16}$
0	-2
1	0
2	4
3	10
4	18



Note. When the integral values do not satisfactorily outline the shape of the curve, fractional values of x must also be used as above.

This graph not only represents the solutions of $x^2 + x - 2 = 0$; but, what is far more important, it represents the values and variation of the polynomial $x^2 + x - 2$ for all values of x (within the limits of the figure). Thus EF represents the value of the trinomial for $x = \frac{1}{2}$, and CD that for $x = -2\frac{1}{4}$.

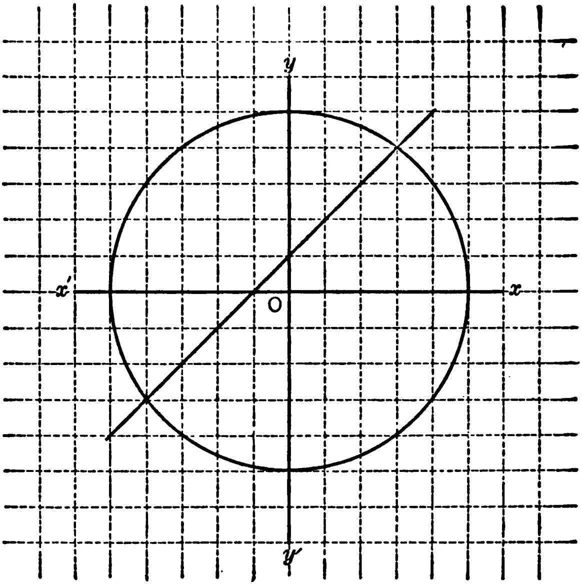
GRAPHS OF SIMULTANEOUS QUADRATIC EQUATIONS

170. PREPARATORY.

1. In the same diagram construct graphs to represent the equations:

$$\begin{cases} x^2 + y^2 = 25, \\ x - y = -1. \end{cases}$$

Compare the result with this figure.



2. Solve the system of equations in Exercise 1.

Compare the values of x and y with the coördinates of the intersections of the graphs.

In how many points does the straight line intersect a circle?

How many solutions has the given system of equations?

WRITTEN EXERCISES

Treat each trinomial of Nos. 1 to 9 below as follows:

(1) Draw the graph of the trinomial.

(2) From the graph discuss the variations of the trinomial as in Section 169.

(3) From the graph read approximately the roots of the equation resulting from equating the trinomial to zero.

(4) If the trinomial is equated to a, read from the graph the range of values of a, for which the roots of the equation are, (i) real and distinct, (ii) real and equal, (iii) imaginary.

(5) Last of all, verify those of the preceding results which relate to roots of the equations by solving the equations.

1. $x^2 + 5x - 6$.	4. $x^2 + 4x - 5$.	7. $x^2 + 2x + 3$.
2. $x^2 + 3x + 2$.	5. $3x^2 - 2x$.	8. $x^2 - 2x - 1$.
3. $2x^2 - 5x + 2$.	6. $x^2 - 4$.	9. $8 + 2x - x^2$.

10. Draw the graph of the function $2x^2 + 5x - 3$, and state how it varies as x varies from a negative value, numerically large at will, through zero to a large positive value. For what values of x is the function positive? For what values negative?

11. Draw the graph of the function $6x^2 - x - 5$, and state how it varies as x varies from a negative value, numerically large at will, through zero to a large positive value. For what values of x is the function positive? For what values negative?

12. What is the graphic condition that $ax^2 + bx + c$ shall have the same sign for all values of x? What must therefore be the character of the roots of $ax^2 + bx + c = 0$, if the trinomial $ax^2 + bx + c$ has the same sign for all values of x?

Determine m so that each of the following trinomials shall be positive for all values of x:

13. $x^2 + mx + 5$. **14.** $3x^2 - 5x + m$. **15.** $mx^2 + 6x + 8$.

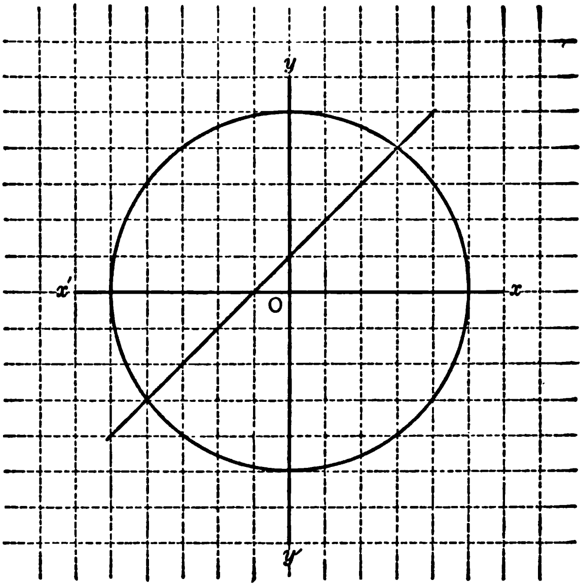
GRAPHS OF SIMULTANEOUS QUADRATIC EQUATIONS

170. Preparatory.

1. In the same diagram construct graphs to represent the equations:

$$\begin{cases} x^2 + y^2 = 25, \\ x - y = -1. \end{cases}$$

Compare the result with this figure.



2. Solve the system of equations in Exercise 1.

Compare the values of x and y with the coördinates of the intersections of the graphs.

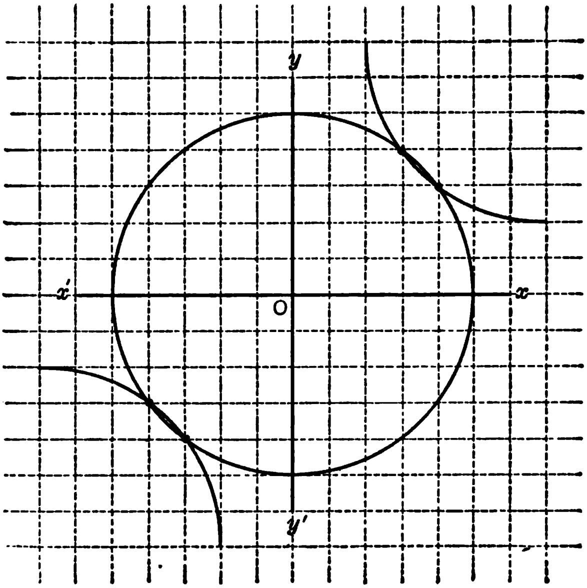
In how many points does the straight line intersect a circle?

How many solutions has the given system of equations?

3. Construct in one diagram the graphs of the equation:

$$\begin{cases} x^2 + y^2 = 25, \\ xy = 12. \end{cases}$$

Compare the result with this figure.



4. Solve the system of equations in Exercise 3.

Compare the values of x and y with the coördinates of the intersections of the graphs.

171. Graphical Solution of Two Simultaneous Equations. Every point of the graph of one of the equations has coördinates, x and y, that satisfy that equation; the points of intersection are points of both graphs and therefore have coördinates, x and y, that satisfy both equations. Hence, to solve two simultaneous equations graphically, draw their graphs and read the coördinates of their intersections. The coördinates of each point of intersection correspond to a solution. If the graphs do not intersect, the system of equations has no real roots, all of them being imaginary numbers.

To find the relation between the graphs of equation (1) and equation (2), they are plotted on the same axes.

In equation (1) y, or $f(x) = x - \sqrt{2x^2 + 17} + 3$.
In equation (2) y, or $f(x) = x^2 - 6x + 8$.

VALUES FOR PLOTTING

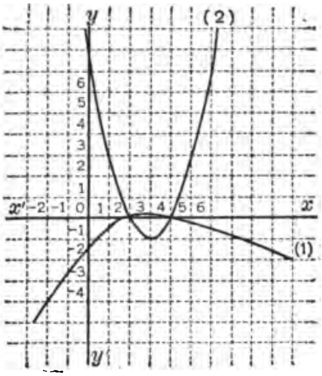
Equation (1)

x	$f(x)$
-2	-4
-1	-2.3
0	-1.1
1	-0.3
2	0
3	0.1
4	0
6	-0.4

Equation (2)

x	$f(x)$
-2	24
-1	15
0	8
1	3
2	0
3	-1
4	0
5	3

In finding values of y always take the positive sign of the radical.



Although the graphs of equations (1) and (2) are entirely different, each cuts the x-axis in points 2 and 4. This shows why both roots of equation (2) also satisfy equation (1). Hence, no *extraneous* roots are introduced in this case by squaring.

The equations not being of higher degree than the second can not have other roots.

2. Given $\sqrt{x-1} - \sqrt{x-5} = \sqrt{2x-6}$. *(1)*

Squaring, $\sqrt{(x-1)(x-5)} = 0$. *(2)*

Squaring, $x^2 - 6x + 5 = 0$. *(3)*

Solving (3), $x = 1, 5$. But 1 is not a root of (1). 5 satisfies equations (1), (2), (3), and 1 satisfies (2) and (3).

indicates that the other two roots are imaginary, for the curve does not rise high enough to cut the axis again, so as to make two more real intersections. The imaginary roots are $-\sqrt{-2}$ and $+\sqrt{-2}$.

Thus we have an illustration of the fact that imaginary roots always occur in pairs. Consequently, a cubic equation can only have two imaginary roots and must have either one or three real roots. An equation of the fourth degree may have either one pair or two pairs of imaginary roots; hence, it may have either none, two, or four real roots.

If the curve just touches the x-axis and then turns away from it, the equation has an even number of *equal* real roots corresponding to this point of contact.

ORAL EXERCISES

1. How many roots has a cubic equation? If the roots of the cubic equation are all real and unequal, in how many points does the graph cut the x-axis?

2. Can a cubic equation have two and only two real roots?

3. If the roots of a cubic equation are 2, 2 and -5, in how many places does its graph cross the x-axis? What will indicate the equal roots?

4. When might the graph of a fourth degree equation not cut the x-axis at all?

5. What is the greatest number of intersections with the x-axis possible for the graph of a fifth degree equation?

173. Graphs of Radical Equations. The graphs of radical equations picture the changes which account for extraneous roots.

EXAMPLES

1. Given $$x-\sqrt{2x^2+17}=-3. \qquad (1)$$

Transposing and squaring, $$x^2-6x+8=0. \qquad (2)$$

Then, $$(x-2)(x-4)=0. \qquad (3)$$

Therefore the roots are 2 and 4, both of which satisfy (*1*).

To find the relation between the graphs of equation (1) and equation (2), they are plotted on the same axes.

In equation (1) y, or $f(x) = x - \sqrt{2x^2 + 17} + 3$.
In equation (2) y, or $f(x) = x^2 - 6x + 8$.

VALUES FOR PLOTTING

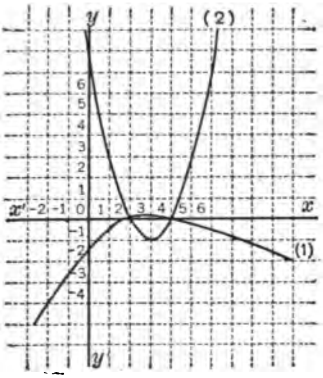
EQUATION (1)

x	$f(x)$
-2	-4
-1	-2.3
0	-1.1
1	-0.3
2	0
3	0.1
4	0
6	-0.4

EQUATION (2)

x	$f(x)$
-2	24
-1	15
0	8
1	3
2	0
3	-1
4	0
5	3

In finding values of y always take the positive sign of the radical.



Although the graphs of equations (1) and (2) are entirely different, each cuts the x-axis in points 2 and 4. This shows why both roots of equation (2) also satisfy equation (1). Hence, no *extraneous* roots are introduced in this case by squaring.

The equations not being of higher degree than the second can not have other roots.

2. Given $\sqrt{x-1} - \sqrt{x-5} = \sqrt{2x-6}$. (1)

Squaring, $\sqrt{(x-1)(x-5)} = 0$. (2)

Squaring, $x^2 - 6x + 5 = 0$. (3)

Solving (3), $x = 1, 5$. But 1 is not a root of (1). 5 satisfies equations (1), (2), (3), and 1 satisfies (2) and (3).

CHAPTER VII

QUADRATIC EQUATIONS

THEORY

175. The **general form** for a quadratic polynomial with one unknown quantity is ax^2+bx+c, where a, b, and c denote any algebraic expressions not involving x, and where a is not zero. If a is zero the polynomial is linear.

For example: 1. $5x^2-7x+8$.

Here $a=5$, $b=-7$, $c=8$.

2. $\frac{7m}{2n+1}x^2+3x-\frac{5m}{2n-1}$.

Here $a=\frac{7m}{2n+1}$, $b=3$, $c=\frac{-5m}{2n-1}$.

WRITTEN EXERCISES

Put the following expressions into the form ax^2+bx+c:

1. $3x+5x(x-2)+4(x^2-5)$.
2. $7(4x-1)+(x+3)(x-2)$.
3. $a(bx+c)(2dx+3e)$.
4. $(x+q)-q(x^2-11)$.
5. $(ax+b)(cx+d)$.
6. $(x^2-a)+(x^2-b)$.
7. $(x+q)(x+p)-(x-q)(2x-p)$.
8. $\left(\frac{x}{2}+\frac{1}{3}\right)^2-\frac{2}{3}\left(\frac{9x}{8}-\frac{16}{3}\right)^2$.
9. $x^2+ab-ax-b(a+x+x^2)$.
10. $x(x-2)(x-4)-x^2(x-5)$.
11. $(2x+1)^2-(3x+1)^2+(4x+1)^2$.
12. $(x-1)(x-2)(x-3)-(x+1)(x+2)(x+3)$.

176. Similarly, every quadratic equation can be put into the form $ax^2 + bx + c = 0$ by transposing all terms to the left member and then putting the polynomial which constitutes the left member into the form $ax^2 + bx + c$.

For example:

Given $(mx + 3a)^2 = mx^2 - 5(amx - 2)$,

then $m^2x^2 + 6amx + 9a^2 = mx^2 - 5amx + 10$,

transposing, $(m^2 - m)x^2 + 11amx + 9a^2 - 10 = 0$.

Hence, $a = m^2 - m$, $b = 11am$, $c = 9a^2 - 10$.

METHODS OF SOLUTION

177. General Solution. By solving the general quadratic equation $ax^2 + bx + c = 0$, general formulas for the roots are obtained.

Solve:
$$ax^2 + bx + c = 0. \qquad (1)$$

Dividing by a, which is not 0,
$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0. \qquad (2)$$

Adding $\frac{b^2}{4a^2}$ to complete the square and subtracting the same,
$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0. \qquad (3)$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0. \qquad (4)$$

Writing the second term as the square of its square root,
$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0. \qquad (5)$$

Factoring (5),
$$\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0. \qquad (6)$$

$$\therefore x = -\frac{b}{2a} + \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right) \text{ and } x = -\frac{b}{2a} - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right). \qquad (7)$$

Denoting these roots by r_1 and r_2:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

By substituting in these formulas the values, including the signs, that a, b, c have in any particular equation, the roots of that equation are obtained. This is called **solution by formula.**

EXAMPLE

Solve: $3x^2 - 9x + 5 = 0.$ (1)

Here, $a = 3,\ b = -9,\ c = 5,$ (2)

and $x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \cdot 3 \cdot 5}}{2 \cdot 3}.$ (3)

$= \frac{9 \pm \sqrt{81 - 60}}{6} = \frac{9 \pm \sqrt{21}}{6}.$ (4)

WRITTEN EXERCISES

Solve by formula:

1. $x^2 - 4x + 4 = 0.$
2. $x^2 - 5x + 6 = 0.$
3. $x^2 - 3x + 2 = 0.$
4. $x^2 - x - 1 = 0.$
5. $x^2 + 3x + 1 = 0.$
6. $x^2 + 2x - 1 = 0.$
7. $x^2 - 13x + 9 = 0.$
8. $2x^2 - 7x - 3 = 0.$
9. $x^2 - x + 6 = 0.$
10. $2x^2 - x + 2 = 0.$
11. $3x^2 - 2x + 1 = 0.$
12. $7x^2 + 6x - 4 = 0.$
13. $4x^2 - 12x + 9 = 0.$
14. $3x^2 + 5x - 2 = 0.$
15. $5x^2 - 4x + 6 = 0.$
16. $7x^2 + 5x - 8 = 0.$

178. Literal Quadratic Equations. When any of the coefficients of a quadratic equation involve letters, the equation is called a **literal quadratic equation.**

Such equations are solved in the usual way.

EXAMPLES

1. Solve: $x^2 + 6mx + 8 = 0.$ (1)

$x^2 + 6mx = -8.$ (2)

Completing the square, $x^2 + 6mx + 9m^2 = 9m^2 - 8.$ (3)

$\therefore (x + 3m)^2 = 9m^2 - 8.$ (4)

$\therefore x + 3m = \pm\sqrt{9m^2 - 8}.$ (5)

$\therefore x = -3m \pm \sqrt{9m^2 - 8}.$ (6)

2. Solve: $t^2 + gt + h = 0.$ (1)

Here, $a = 1,\ b = g,\ c = h.$ (2)

Hence, by Sec. 177, $t = \frac{-g \pm \sqrt{g^2 - 4h}}{2}.$ (3)

3. Solve: $$gt^2+2\,vt=2\,s. \qquad (1)$$

$$gt^2+2\,vt-2\,s=0. \qquad (2)$$

Here, $a=g,\ b=2\,v,\ c=-2\,s.$ (3)

Hence, by Sec. 177, $$t=-\frac{2\,v}{2\,g}\pm\frac{1}{2\,g}\sqrt{(2\,v)^2-4\,g(-2\,s)} \qquad (4)$$

$$=-\frac{v}{g}\pm\frac{1}{g}\sqrt{v^2+2\,gs}. \qquad (5)$$

$$=\frac{1}{g}(-v\pm\sqrt{v^2+2\,gs}). \qquad (6)$$

WRITTEN EXERCISES

Solve:

1. $t^2-8\,t+24\,d=9\,d^2.$
2. $ay^2-(a-b)y-b=0.$
3. $w^2+4\,aw+a^2=0.$
4. $v^2-4\,amv=(a^2-m^2)^2.$
5. $w^2-a^2=2\,b(a-w).$
6. $t^2+at=k.$
7. $u^2+ku+1=0.$
8. $v^2+mv=1.$
9. $ax^2+bx+c=0.$
10. $x^2+ax+b=0.$
11. $m^2x^2+2\,mx=-1.$
12. $x^2+2\,px-1=0.$
13. $4\,x^2-4\,ax+16=0.$
14. $a^2x^2+2\,ax+5=0.$
15. $m^2x^2+4\,mx-6=0.$
16. $x^2-4\,ax=9.$
17. $5\,ax^2+3\,bx+2\,b^3=0.$
18. $b^2x^2-2\,bx=ac-1.$
19. $x^2-3\,ax+10\,a^2=0.$
20. $2\,x^2-3\,x=a(3-4\,x).$

179. Collected Methods. We have used three methods of solving quadratic equations:

1. *Factoring.*

EQUATION	FACTORS	ROOTS
$x^2-3\,x+2=0.$	$(x-2)(x-1).$	$x=2,\ x=1.$
$x^2-(a+b)x+ab=0.$	$(x-a)(x-b).$	$x=a,\ x=b.$

2. *Completing the square.*

EQUATION	SOLUTION	ROOTS
$x^2+x+2=0.$	See Sec. 371, Part I.	$x=\frac{-1\pm\sqrt{-7}}{2}.$

$ax^2+bx+c=0.$	See Sec. 371, Part I.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$

3. *Formula.*

Equation	Solution	Roots
$3x^2+2x-7=0.$	See Sec. 177.	$x=\frac{-1\pm\sqrt{22}}{3}.$
$ax^2+bx+c=0.$	See Sec. 177.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

WRITTEN EXERCISES

Solve by factoring:

1. $x^2-x-6=0.$
2. $x^2-x-2=0.$
3. $x^2+x-2=0.$
4. $x^2+x-6=0.$
5. $x^2+3x+2=0.$
6. $x^2-x-30=0.$
7. $x^2+x-12=0.$
8. $x^2-3x+2=0.$
9. $x^2+11x+30=0.$
10. $x^2-7x+12=0.$

Solve by completing the square:

11. $x^2+x+1=0.$
12. $x^2+3x+1=0.$
13. $x^2-\frac{1}{2}x+1=0.$
14. $x^2-1.5x+.5=0.$
15. $x^2-5x+10=0.$
16. $x^2-16x+60=0.$
17. $x^2+\frac{3}{4}x+\frac{1}{4}=0.$
18. $x^2+7.5x-3.5=0.$

Solve by formula:

19. $3x^2+x+5=0.$
20. $2x^2-5x-3=0.$
21. $4x^2+3x-1=0.$
22. $5x^2+2x+6=0.$
23. $x^2+x+1=0.$
24. $x^2+1=0.$
25. $x^2+15x+56=0.$
26. $x^2+8x+33=0.$
27. $x^2-10x+34=0.$
28. $2x^2+3x-27=0.$

Solve and test, using whichever of the methods in Sec. 179 seems most convenient:

29. $9y^2-4=0.$
30. $6x^2-13x+6=0.$
31. $5x^2-4x+4=0.$
32. $t^2+11t+30=0.$

33. $6s^2 - 5s - 6 = 0.$

34. $6r^2 - 2r - 4 = 0.$

35. $w^2 + 4w - 3 = 0.$

36. $\dfrac{2x-1}{2x+1} + \dfrac{2x+1}{2x-1} = 3.$

37. $x^2 - 2x + 3 = 0.$

38. $x^2 - 0.5x + 0.06 = 0.$

39. $x^2 - .7x + .12 = 0.$

40. $11x^2 + 1 = 4(2 - x)^2.$

41. $x^2 + (a + b)x + ab = 0.$

42. $x^2 - (b + c)x + bc = 0.$

43. $2ax + (a - 2)x - 1 = 0.$

44. $\dfrac{a^2}{b+x} - \dfrac{a^2}{b-x} = c.$

45. The product of two consecutive positive integers is 306. Find the integers.

Solution.

1. Let x be the smaller integer.
2. Then $x + 1$ is the larger.
3. $\therefore x(x + 1)$ is their product.
4. $\therefore x(x + 1) = 306$, by the given conditions.
5. $\therefore x^2 + x - 306 = 0$, from (4).
6. $\therefore x = \dfrac{-1 \pm \sqrt{1 + 1224}}{2} = \dfrac{-1 \pm 35}{2} = 17$, or -18, solving (5).

Since the integers are to be positive, the value -18 is not admissible. $x = 17$, $\therefore x + 1 = 18$, and the integers are 17 and 18.

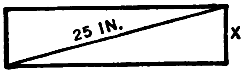
Test. $17 \cdot 18 = 306.$

46. There is also a pair of consecutive negative integers whose product is 306. What are they?

47. If the square of a certain number is diminished by the number, the result is 72. Find the number.

48. A certain number plus its reciprocal is -2. What is the number?

49. A certain positive number minus its reciprocal is $\frac{5}{6}$. What is the number? What negative number has the same property?



50. The perimeter of the rectangle shown in the figure is 62 in. Find the sides.

51. One perpendicular side of a certain right triangle is 31 units longer than the other; the square of their sum exceeds the square of the hypotenuse by 720. Find the sides.

52. In a right triangle of area 60 sq. ft.; the difference between the perpendicular sides is 7. Find the three sides.

Note. Those who have studied geometry may take up some of the problems based upon geometric properties found in Chapter X.

RELATIONS BETWEEN ROOTS AND COEFFICIENTS

180. Relation of Roots to Coefficients. By adding the values found for the roots (Sec. 177), we obtain

$$r_1 + r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}.$$

Multiplying the values, we find

$$r_1 r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \times \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}.$$

Applying these results to the equation $x^2 + px + q = 0$, we have:

$$r_1 + r_2 = -p,$$
$$r_1 r_2 = q.$$

In words:

In the equation $x^2 + px + q = 0$, *the coefficient of* x *with its sign changed is the sum of the roots, and the absolute term is their product.*

Every quadratic equation can be put into the form $x^2 + px + q = 0$ by dividing both members by the coefficient of x^2.

181. Symmetric Functions of the Roots. It is apparent that the equations in Section 180 remain unchanged if r_1 and r_2 are interchanged. On this account the expressions $r_1 + r_2$ and $r_1 r_2$ are called symmetric functions of the roots of the quadratic equation. There are other such functions, but these only will be treated here. The two following sections show some of their uses.

182. By means of Section 180 a quadratic equation may be written whose roots are any two given numbers.

EXAMPLES

1. Write an equation whose roots are 2, -3.

$$-p = r_1 + r_2 = 2 + (-3) = -1. \quad \therefore p = 1.$$
$$q = r_1 r_2 = 2(-3) = -6.$$
$$\therefore x^2 + x - 6 = 0 \text{ is the equation sought.}$$

2. Write an equation whose roots are $\frac{1}{2} + \sqrt{-3}$, $\frac{1}{2} - \sqrt{-3}$.

$$-p = r_1 + r_2 = (\tfrac{1}{2} + \sqrt{-3}) + (\tfrac{1}{2} - \sqrt{-3}) = 1. \quad \therefore p = -1.$$
$$q = r_1 r_2 = (\tfrac{1}{2} + \sqrt{-3})(\tfrac{1}{2} - \sqrt{-3}) = \tfrac{1}{4} - (-3) = \tfrac{13}{4}.$$
$$\therefore x^2 - x + \tfrac{13}{4} = 0 \text{ is the equation sought.}$$

WRITTEN EXERCISES

Write the equations whose roots are:

1. 4, 5.
2. $\frac{3}{4}$, $\frac{4}{3}$.
3. 7, $-1\frac{3}{7}$.
4. -4, $+4$.
5. $\frac{2}{3} \pm \sqrt{-5}$.
6. 24, 30.
7. $8\frac{3}{5}$, 10.
8. -5, -20.
9. $-\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.
10. $\frac{5}{6} \pm \frac{1}{6}\sqrt{-47}$.
11. a, $-b$.
12. 8, -40.
13. $a - bi$, $a + bi$.
14. $1 + 2i$, $1 - 2i$.
15. $\frac{1}{2} - \sqrt{2}$, $\frac{1}{2} + \sqrt{2}$.

183. **Testing Results.** The ultimate test of the correctness of a solution is that of substitution; but this is not always convenient, especially when the roots are irrational. In such cases, the relations between the roots and coefficients are of use.

For example; Solving $2x^2 - 5x + 6 = 0$,

or $x^2 - \frac{5}{2}x + 3 = 0$,

the roots are $r_1 = \frac{5}{4} + \frac{1}{4}\sqrt{-23}$ and $r_2 = \frac{5}{4} - \frac{1}{4}\sqrt{-23}$.

Adding, $-(r_1 + r_2) = -\frac{10}{4} = -\frac{5}{2}$, the coefficient of x.

Multiplying, $r_1 r_2 = (\frac{5}{4})^2 - (\frac{1}{4}\sqrt{-23})^2 = \frac{25}{16} + \frac{23}{16} = 3$, the absolute term.

Therefore, the roots are correct. (Sec. 180.)

184. In what follows, the coefficients a, b, c are restricted to rational numbers.

185. Character of the Roots. By examining the formula for the roots, $-\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$, it appears that the character of the roots as real or imaginary, rational or irrational, equal or unequal, depends upon the value of the expression $b^2 - 4ac$.

1. If $b^2 - 4ac$ is positive, the roots are real.

Thus, in $x^2 + 4x - 3 = 0$, $b^2 - 4ac = 16 + 12$, or 28, ∴ the roots are real and unequal.

2. If $b^2 - 4ac$ is a perfect square, the indicated square root can be extracted, and the roots are rational.

Thus, in $x^2 - 4x + 3 = 0$, $b^2 - 4ac = 16 - 12$, or 4, ∴ the roots are rational and unequal.

3. If $b^2 - 4ac$ is not a perfect square, the indicated root cannot be extracted and the roots are irrational.

Thus, in $x^2 + 5x + 1 = 0$, $b^2 - 4ac = 25 - 4 = 21$, ∴ the roots are irrational.

4. If $b^2 - 4ac = 0$, the radical is zero, and the two roots are equal.

Thus, in $x^2 - 10x + 25 = 0$, $b^2 - 4ac = 100 - 4 \cdot 25 = 0$, ∴ the roots are equal.

5. If $b^2 - 4ac$ is negative, the roots are imaginary.

Thus, in $2x^2 - x + 1 = 0$, $b^2 - 4ac = 1 - 8$, or -7, ∴ the roots are complex numbers.

Consequently, it is merely necessary to calculate $b^2 - 4ac$ to know in advance the nature of the roots of a quadratic equation.

186. Discriminant. Because its value determines the character of the roots, the expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

ORAL EXERCISES

Without solving the equations, find the nature of the roots of:

1. $x^2 + x - 20 = 0$.
2. $x^2 + x - 3 = 0$.
3. $2x^2 - x + 2 = 0$.
4. $3x^2 - x + 3 = 0$.
5. $2x^2 + 2x - 4 = 0$.
6. $5x^2 - 3x + 6 = 0$.
7. $3x^2 - 4x + 5 = 0$.
8. $6x^2 + x - 1 = 0$.
9. $7x^2 + 3x - 4 = 0$.
10. $-4x + 8x^2 + 1 = 0$.
11. $5 + 4x^2 - 3x = 0$.
12. $7x + 6 + x^2 = 0$.
13. $-6x + 9x^2 + 3 = 0$.
14. $x^2 - 6x + 4 = 0$.

187. The relation $x^2+px+q=x^2-(r_1+r_2)x+r_1r_2$ may be written:

$$(1) \qquad x^2+px+q=(x-r_1)(x-r_2).$$

And since $x^2+px+q=\frac{ax^2+bx+c}{a}$ in which $p=\frac{b}{a}$, $q=\frac{c}{a}$, we have

$$(2) \qquad ax^2+bx+c=a(x-r_1)(x-r_2).$$

188. The solution of a quadratic equation, therefore, enables us to factor every polynomial of either form (1) or (2).

Since r_1 and r_2 involve radicals:

1. *The factors will generally be irrational.*

2. *The factors will be rational when* r_1 *and* r_2 *are so; that is when* b^2-4ac *is a perfect square.*

3. *The two factors involving* x *will be the same when the roots are equal; that is, when* $b^2-4ac=0$.

In the last case the expressions are squares and

(1) becomes $(x-r_1)^2$, and
(2) becomes $[\sqrt{a}(x-r_1)]^2$.

EXAMPLES

	TRINOMIAL	b^2-4ac	NATURE OF FACTORS
1.	$3x^2-7x+2$	$49-4\cdot3\cdot2=25$	rational of 1st degree.
2.	$3x^2-7x+3$	$49-4\cdot3\cdot3=13$	irrational.
3.	$2x^2-8x+8$	$64-4\cdot2\cdot8=0$	equal.

ORAL EXERCISES

By means of the above test, select the squares; also the trinomials with rational factors of the first degree:

1. $8x^2-8x+2$.
2. $\frac{y^2}{3}+4y+12$.
3. $3x^2+3x+1$.
4. $3z^2+2z+12$.
5. x^2+3x-2.
6. $a^2x^2+2ax+1$.
7. $4x^2+4x+1$.
8. $x^2-8x+15$.
9. $6x^2+5x-4$.
10. $6x^2-5x+9$.
11. $4x^2-4x-3$.
12. $8x^2-9x+3$.

189. General Method of Factoring Quadratic Trinomials. The actual factors of any quadratic trinomial of the form $ax^2 + bx + c$ can be found by solving the quadratic equation:

$$ax^2 + bx + c = 0,$$

and substituting the roots in the relation:

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

EXAMPLES

1. Factor: $$6x^2 + 5x - 4. \qquad (1)$$

Solving $6x^2 + 5x - 4 = 0$, $$x = -\tfrac{4}{3}, \qquad (2)$$ $$x = \tfrac{1}{2}.$$

Substituting $-\frac{4}{3}$ for r_1, $\frac{1}{2}$ for r_2, and 6 for a. $$a(x - r_1)(x - r_2) = 6\left(x + \tfrac{4}{3}\right)\left(x - \tfrac{1}{2}\right). \qquad (3)$$

Therefore, $$6x^2 + 5x - 4 = 6\left(x + \tfrac{4}{3}\right)\left(x - \tfrac{1}{2}\right). \qquad (4)$$

Using the factor 6 as $3 \cdot 2$, the result may be written $(3x + 4)(2x - 1)$.

2. Factor: $$x^2 - x + 1. \qquad (1)$$

Solving $x^2 - x + 1 = 0$, $$x = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}. \qquad (2)$$ $$x = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

Substituting these values of x for r_1 and r_2,

$$a(x - r_1)(x - r_2) = \left(x - \frac{1 + \sqrt{-3}}{2}\right)\left(x - \frac{1 - \sqrt{-3}}{2}\right). \qquad (3)$$

Thus, we have a general method of factoring which does not depend upon the trial methods of Chapter XII, Part I.

WRITTEN EXERCISES

Factor:

1. $3x^2 - 2x - 5$.
2. $9x^2 - 3x - 6$.
3. $6y^2 + y - 1$.
4. $15y^2 - 4y - 35$.
5. $10w^2 - 12w + 2$.
6. $9v^2 - 17v - 2$.
7. $6x^2 + 25x + 14$.
8. $2z^2 + 5z + 2$.
9. $6x^2 - 7x + 3$.
10. $5x^2 - 40x + 6$.
11. $a^{2m} - 2a^m - 3$.
12. $c^4 - 13c^2 + 36$.

RADICAL EQUATIONS

190. The simpler forms of radical equations can be solved by squaring both members, but other forms require special methods.

EXAMPLE

Solve:
$$5x^2-3x+\sqrt{5x^2-3x+2}=18. \tag{1}$$

Putting $5x^2-3x=y$, the given equation becomes
$$y+\sqrt{y+2}=18. \tag{2}$$

Subtracting y,
$$\sqrt{y+2}=18-y. \tag{3}$$

Squaring,
$$y+2=324-36y+y^2. \tag{4}$$

Rearranging,
$$y^2-37y+322=0. \tag{5}$$

Solving,
$$y=\frac{37\pm\sqrt{37^2-4\cdot 322}}{2} \tag{6}$$
$$=23 \text{ or } 14. \tag{7}$$

Hence the values of x are determined from
$$5x^2-3x=23, \tag{8}$$

and
$$5x^2-3x=14. \tag{9}$$

Solving these,
$$x=\frac{3\pm\sqrt{469}}{10}, \tag{10}$$

and
$$x=2, \text{ or } -1.4. \tag{11}$$

Test. Substituting, it appears that 2 and -1.4 satisfy the equation. The values $\frac{3\pm\sqrt{469}}{10}$ satisfy
$$5x^2-3x-\sqrt{5x^2-3x+2}=18.$$

191. Radical equations in quadratic form are more easily handled in the notation of exponents.

EXAMPLE

Solve:
$$x^{-\frac{1}{2}}+x^{-1}-6=0. \tag{1}$$

Rearranging,
$$x^{-1}+x^{-\frac{1}{2}}-6=0. \tag{2}$$

Factoring,
$$(x^{-\frac{1}{2}}+3)(x^{-\frac{1}{2}}-2)=0. \tag{3}$$

Therefore,
$$x^{-\frac{1}{2}}=-3, \text{ and } x^{-\frac{1}{2}}=2. \tag{4}$$

Or,
$$x=\tfrac{1}{9}, \text{ and } x=\tfrac{1}{4}. \tag{5}$$

Substituting in step (1), $\frac{1}{4}$ satisfies the equation, but $\frac{1}{9}$ is an extraneous root.

192. If the factors of a quadratic form are not readily apparent, the quadratic formula may be applied.

193. If the equation contains denominators irrational in the unknowns, it is generally best to clear of fractions.

WRITTEN EXERCISES

Solve:

1. $6x - x^{\frac{1}{2}} - 12 = 0.$

2. $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 6 = 0.$

3. $x^2 + x + \sqrt{x^2 + x + 1} = -1.$

4. $x^2 + a + \sqrt{x^2 + 2a} = -a.$

5. $x^2 + 6x - 5 = \sqrt{x^2 + 6x + 7}.$

6. $\sqrt{x^2 + 6x - 16} + (x + 3)^2 = 25.$

7. $2x - 5x^{\frac{1}{2}} + 2 = 0.$

8. $x^{-\frac{2}{3}} - x^{-\frac{1}{3}} - 6 = 0.$

9. $3x^{-\frac{1}{2}} - 4x^{-\frac{1}{4}} = 7.$

10. $x^2 + 3x - 3 + \sqrt{x^2 + 3x + 17} = 0.$

11. $x + \sqrt{x^2 - 8} = \dfrac{3x - \sqrt{x^2 - 8}}{x - \sqrt{x^2 - 8}}.$

12. $\sqrt{x(a + x)} + \sqrt{x(a - x)} = 2\sqrt{ax}.$

13. $\sqrt{(\sqrt{1 + x^2} + x) \div (\sqrt{1 + x^2} - x)} = 4.$

14. $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}.$

15. $\sqrt{2x + 11} - \sqrt{x - 3} - \sqrt{x + 2} = 0.$

16. $\sqrt{y + 1} + \sqrt{y - 2} = \sqrt{2y + 3}.$

17. $\sqrt{\dfrac{x}{c} + \dfrac{a}{b}} + \sqrt{\dfrac{x}{c} - \dfrac{a}{b}} = \sqrt{\dfrac{4x}{c} - \dfrac{2a}{b}}.$

18. $\sqrt{2y - 2} + \sqrt{y} = \sqrt{6y - 5}.$

19. $2a = \sqrt{2ax + 5a^2} - \sqrt{2ax - 3a^2}.$

20. $\dfrac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}} - 1 = 0.$

CERTAIN HIGHER EQUATIONS SOLVED BY THE AID OF QUADRATIC EQUATIONS

194. We have found the general solution of linear and quadratic equations with one unknown. Equations of the third and the fourth degree can also be solved generally by algebra, and certain types of equations of still higher degree as well; but these solutions do not belong to an elementary course. We shall take up only certain equations of higher degree whose solution is readily reduced to that of quadratic equations.

EXAMPLES

1. Solve: $x^6 - 3x^3 - 4 = 0.$ (*1*)

Let $y = x^3$, then $y^2 - 3y - 4 = 0.$ (*2*)
Solving (*2*), $y = 4,$
and $y = -1.$ (*3*)
∴ by the substitution in (*2*), $x^3 = 4$, or $x^3 - 4 = 0,$
and $x^3 = -1$, or $x^3 + 1 = 0.$ (*4*)
Factoring (*4*), $x^3 - 4 = (x - \sqrt[3]{4})(x^2 + \sqrt[3]{4} \cdot x + \sqrt[3]{4^2}) = 0,$ (*5*)
and $x^3 + 1 = (x + 1)(x^2 - x + 1) = 0.$ (*6*)
Solving (*5*), $x = \sqrt[3]{4}$, or $\sqrt[3]{4}(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})$, or $\sqrt[3]{4}(-\frac{1}{2} - \frac{1}{2}\sqrt{-3})$. (*7*)
Solving (*6*), $x = -1$, or $\frac{1}{2} - \frac{1}{2}\sqrt{-3}$, or $\frac{1}{2} + \frac{1}{2}\sqrt{-3})$. (*8*)

2. Solve: $(x^2 - 3x + 1)(x^2 - 3x + 2) = 12.$ (*1*)

This may be written $(x^2 - 3x + 1)[(x^2 - 3x + 1) + 1] = 12.$ (*2*)
If y is put for $x^2 - 3x + 1$, the equation becomes $y(y + 1) = 12,$ (*3*)
or $y^2 + y - 12 = 0.$ (*4*)
Solving, $y = 3$ or $-4.$ (*5*)
Then from (*3*), $x^2 - 3x + 1 = 3,$ (*6*)
and $x^2 - 3x + 1 = -4.$ (*7*)
Solving (*6*), $x = \frac{3 \pm \sqrt{17}}{2}.$ (*8*)
Solving (*7*), $x = \frac{3 \pm \sqrt{-11}}{2}.$ (*9*)

These are the four roots of the given equation of the fourth degree.

WRITTEN EXERCISES

Solve as above:

1. $x^4 - 3x^2 + 1 = 0.$
2. $12 - x^4 = 11x^2.$
3. $x^4 + ax^2 - 8a^2 = 0.$
4. $x^2 + 5 = \dfrac{5}{x^2+3}.$
5. $x^6 - 7x^3 + 6 = 0.$
6. $x^8 - 3x^4 + 2 = 0.$
7. $x^{10} - 5x^5 + 6 = 0.$
8. $x^4 + 13x^2 + 36 = 0.$
9. $ax^{2n} - bx^n + c = 0.$
10. $\dfrac{1}{x^2+1} + \dfrac{1}{x^2+2} = \dfrac{1}{x^2+3}.$
11. $x^{2n} - 4x^n - 5 = 0.$
12. $2x^6 + 5x^3 + 2 = 0.$
13. $(x^2+4)^2 - 4(x^2+4) + 4 = 0.$
14. $x^2 + 3x = 1 - \dfrac{1}{x^2+3x+1}.$
15. $(x^2 - 3x + 1)(x^2 - 3x + 2) = 12.$
16. $(x^2 + 5x - 1)(x^2 + 5x + 1) = -1.$

195. Binomial Equations. Equations of the form $x^n \pm a = 0$ are called **binomial equations.** The simpler cases admit of being solved by elementary processes.

EXAMPLES

1. Solve: $x^3 - 1 = 0$, or $x^3 = 1$. *(1)*

Factoring $x^3 - 1$, $(x-1)(x^2+x+1) = 0.$ *(2)*

Finding equations equivalent to (2), $x = 1,\ x^2 + x + 1 = 0.$ *(3)*

Solving (3) $x = 1,\ x = \dfrac{-1 \pm \sqrt{-3}}{2}.$ *(4)*

Thus we have found the three numbers such that the cube of each is 1, or *the three cube roots of unity.*

Verify this statement by cubing each number in step *(4)*.

Note that $(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})^2 = \dfrac{-1 - \sqrt{-3}}{2} = \dfrac{-1 - i\sqrt{3}}{2},$

also that $(-\frac{1}{2} - \frac{1}{2}\sqrt{-3})^2 = \dfrac{-1 + \sqrt{-3}}{2} = \dfrac{-1 + i\sqrt{3}}{2}.$

Hence, if ω denotes one of the complex cube roots of 1, ω^2 is the other.

Every number has three cube roots; for example, the cube roots of 8 are 2, and 2ω, and $2\omega^2$.

2. Solve: $x^4+1=0$, or $x^4=-1$. *(1)*

Factoring x^4+1, $(x^2-i)(x^2+i)=0$. *(2)*

Solving *(2)*, $x^2=i$, $x^2=-1$. *(3)*

Solving *(3)*, $x=\pm\sqrt{i}$, $x=\pm\sqrt{-i}$. *(4)*

These are the four numbers, each of which raised to the fourth power equals -1 or *the four fourth roots of* -1.

WRITTEN EXERCISES

1. Find the 3 cube roots of -1 by solving $x^3+1=0$.

2. Find the 4 fourth roots of 1 by solving $x^4-1=0$.

3. Find the 6 sixth roots of 1 by solving

$$(x^6-1)=(x^3-1)(x^3+1)=0.$$

4. Find the 4 fourth roots of 16 by solving $x^4-16=0$.

5. Find the 3 cube roots of 8 by solving $x^3-8=0$.

6. Show that the square of either irrational cube root of -1 is the negative of the other irrational cube root.

7. Show that the sum of the three cube roots of unity is zero; also that the sum of the 6 sixth roots of unity is zero.

SIMULTANEOUS QUADRATIC EQUATIONS WITH TWO UNKNOWNS

196. The simpler cases in which one equation is linear or homogeneous of the second degree, or in which the equations are symmetric, have been solved in Chapter XXIII, Part I. The following Sections 197, 198 treat of special cases, showing how these may be reduced to the simpler systems.

A system of two simultaneous quadratic equations whose terms are of the second degree in x *and* y *with the exception of the absolute terms can be solved by reducing to a system in which one equation is entirely homogeneous.* (Sec. 120, Part I.)

This can be done in two ways:

1. *Make their absolute terms alike, and subtract. The resulting equation has every term of the second degree in* x *and* y.

EXAMPLE

Solve: $\begin{cases} x^2 + xy = 66, & (1) \\ x^2 - y^2 = 11. & (2) \end{cases}$

Multiplying (*2*) by 6, $6x^2 - 6y^2 = 66.$ (*3*)

Subtracting (*1*) from (*3*), $5x^2 - xy - 6y^2 = 0.$ (*4*)

Factoring (*4*), $(5x - 6y)(x + y) = 0.$ (*5*)

Expressing x in terms of y, $x = \frac{6}{5}y$ and $x = -y.$ (*6*)

Substituting $x = \frac{6}{5}y$ in (*2*), $\frac{36}{25}y^2 - y^2 = 11.$ (*7*)

Solving (*7*), $y = \pm 5.$ (*8*)

Substituting $y = \pm 5$ in $x = \frac{6}{5}y$, $x = \pm 6.$ (*9*)

Similarly, substituting $x = -y$ in (*2*), $y^2 - y^2 = 11.$ (*10*)

But this leads to $0 = 11.$ (*11*)

(*11*) being impossible, the solution is

$x = \pm 6,\ y = \pm 5.$ (*12*)

Test. Taking the values to be both positive or both negative, $\left\{\begin{array}{l} (\pm 6)^2 + (\pm 6)(\pm 5) = 66. \\ (\pm 6)^2 - (\pm 5)^2 = 11. \end{array}\right.$

2. *Substitute* vx *for* y *throughout the equations and solve for* v.

EXAMPLE

Solve: $\begin{cases} 2x^2 - 3xy + y^2 = 4, & (1) \\ x^2 - 2xy + 3y^2 = 9. & (2) \end{cases}$

Putting $y = vx$ in (*1*) and (*2*), $2x^2 - 3vx^2 + v^2x^2 = 4,$ (*3*)

and, $x^2 - 2vx^2 + 3v^2x^2 = 9.$ (*4*)

Factoring (*3*) and (*4*), $x^2(2 - 3v + v^2) = 4.$ (*5*)

$x^2(1 - 2v + 3v^2) = 9.$ (*6*)

Equating the values of x^2 in (*5*) and (*6*), $\frac{4}{2 - 3v + v^2} = \frac{9}{1 - 2v + 3v^2}.$ (*7*)

Clearing (*7*) of fractions, $3v^2 + 19v - 14 = 0.$ (*8*)

Solving (*8*), $v = -7,\ \frac{2}{3}.$ (*9*)

Since $y = vx$, $y = -7x,$ (*10*)

and, $y = \frac{2x}{3}.$ (*11*)

From (*1*), $2x^2 + 21x^2 + 49x^2 = 72x^2 = 4.$ (*12*)

Solving (*12*), $x = \pm\frac{1}{3\sqrt{2}}.$ (*13*)

From (*10*), $y = \mp\frac{7}{3\sqrt{2}}.$ (*14*)

Similarly, from (*11*) and (*1*), $x = \pm 3,$ (*15*)

From (*15*) and (*11*), $y = \pm 2.$ (*16*)

Test as usual.

WRITTEN EXERCISES

Solve:

1. $x^2+y^2=13$,
$xy=6$.

2. $x^2-xy=54$,
$xy-y^2=18$.

3. $x^2+xy=12$,
$y^2+xy=24$.

4. $4x^2+3y^2=43$,
$3x^2-y^2=3$.

5. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{5}{4}$,
$x^2+y^2=20$.

6. $x^2+y^2=41$,
$xy=20$.

7. $x^2+xy+y^2=7$,
$3x^2-xy=1$.

8. $x^2+xy-2y^2=4$,
$x^2-3xy+2=0$.

9. $x^2+6xy+2y^2=133$,
$x^2-y^2=16$.

10. $2x^2+3xy+y^2=80$,
$xy+x^2=6$.

11. $x^2-4y^2=20$,
$xy=12$.

12. $4x^2+3y^2=43$,
$3x^2-y^2-3=0$.

197. The substitution of a single variable for a function is the most successful means of simplifying quadratic systems that do not yield to the methods already given.

EXAMPLE

$$\left\{\begin{array}{r} x-y-\sqrt{x-y}=2, \qquad (1)\\ x^3-y^3=2044. \qquad (2)\end{array}\right.$$

Put $\sqrt{x-y}=z$. (3)

From (1), $z^2-z-2=0$.

$$(z-2)(z+1)=0. \qquad (4)$$

$$z=2, \text{ or } -1. \qquad (5)$$

∴ from (3), $x-y=4$, or 1. (6)

From (2),

$$(x-y)(x^2+xy+y^2)=2044. \qquad (7)$$

From (7) and (6),

$$x^2+xy+y^2=2044, \text{ or } 511. \qquad (8)$$

From (6), $y=x-1$. Put this in (8).

Then, $x^2+x(x-1)+(x-1)^2=2044$. (9)

$$x^2+x^2-x+x^2-2x+1=2044. \qquad (10)$$

$$x^2-x-681=0. \qquad (11)$$

Solving (11), $x=\dfrac{1\pm 5\sqrt{109}}{2}$ and $y=\dfrac{-5\pm\sqrt{109}}{2}$.

From (*6*), $y = x - 4$. Put this in (*8*).

$$x^2 + x(x-4) + (x-4)^2 = 511. \qquad (13)$$
$$3x^2 - 12x - 495 = 0. \qquad (14)$$
$$x^2 - 4x - 165 = 0. \qquad (15)$$
$$x = \frac{4 \pm \sqrt{16 + 4.165}}{2} \qquad (16)$$
$$= \frac{4 \pm 2\sqrt{4 + 165}}{2} \qquad (17)$$
$$= 2 \pm 13 \qquad (18)$$
$$= 15, \text{ or } -11. \qquad (19)$$
$$\therefore y = 11, \text{ or } -15. \qquad (20)$$

TEST by substitution.

WRITTEN EXERCISES

Solve:

1. $x^2 - y^2 = 3,$
$x^2 + y^2 - xy = 3.$

2. $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{5}{2},$
$x^2 + y^2 = 90.$

3. $x^2 + xy + y^2 = 7,$
$x^4 + x^2y^2 + x^4 = 21.$

4. $x^2 + y^2 - 1 = 2xy,$
$xy(xy + 1) = 8190.$

5. $x^2 + xy = \frac{5}{12},$
$xy + y^2 = \frac{5}{18}.$

6. $x^2 + 2xy = 55,$
$2x^2 - xy = 35.$

7. $2x^2 + xy = 24,$
$x^2 - y^2 = 5.$

8. $x^2 - y^2 = 8\frac{3}{4},$
$xy + y^2 = \frac{7}{4}.$

9. $x^2 - y^2 = 45,$
$xy + y^2 = 18.$

10. $x - 2y = 2,$
$x^2 - 6y^2 = 14 - xy.$

11. $x^2 + y^2 = 25,$
$x + y = 1.$

12. $x - y = 2,$
$x^2y - xy^2 = 30.$

13. $x^2 + y^2 + 2(x + y) = 12,$
$xy - (x + y) = 2.$

14. $x^2 + 4y^2 = 13,$
$x + 2y = 5.$

15. $2x + y = 5,$
$3x^2 - 7y^2 = 5.$

16. $x - y = 5,$
$\sqrt{x} - \sqrt{y} = 1.$

17. $x^2 + xy + y^2 = 21,$
$x - \sqrt{xy} + y = 3.$

SUGGESTION. In Ex. 17 divid (1) by (2), obtaining

$$x + y + \sqrt{xy} = 7.$$

Add this equation to (2) and find x in terms of y.

18. $x^2 + tx + t^2 = 133,$
$t + x - \sqrt{tx} = 7.$

19. $x^2 + 2xy + 7y^2 = 24,$
$2x^2 - xy - y^2 = 8.$

20. $x(x - y) = 0,$
$x^2 + 2xy + y^2 = 9.$

21. $\sqrt{x} - \sqrt{y} = 2,$
$(\sqrt{x} - \sqrt{y})\sqrt{xy} = 30.$

22. $(2x - 3)(3y - 2) = 0,$
$4x^2 + 12xy - 3y^2 = 0.$

Suggestion. In Ex. 22, from equation (1), $x = \frac{3}{2}$. The corresponding value of y is found by substituting this value of x in equation (2).

23. Two men, A and B, dig a trench in 20 days. It would take A alone 9 days longer to dig it than it would B. How long would it take A and B each working alone?

Note. Those who have studied geometry and are ready for problems based upon geometric properties will find them in Chapter X.

REVIEW

WRITTEN EXERCISES

Solve:

1. $x^2 = 6x - 5.$

2. $w^2 - w - 1 = 0.$

3. $x^2 - 6x - 7 = 0.$

4. $v^2 + 2v + 6 = 0.$

5. $x^2 - 5x + 1 = 0.$

6. $2x^2 - x + 3 = 0.$

7. $3x^2 - x + 7 = 0.$

8. $x^2 - 5x + 11 = 0.$

9. $x^2 - 14x + 5 = 0.$

10. $3x^2 - 9x - \frac{5}{2} = 0.$

11. $x^2 - 4x - 45 = 0.$

12. $x^2 - 4x + 45 = 0.$

13. $x^2 + 4x - 45 = 0.$

14. $x^2 + 4x + 45 = 0.$

15. $x^2 + 30x + 221 = 0.$

16. $x^2 - 30x - 221 = 0.$

17. $x^2 = 6x + 16.$

18. $24 - 10x = x^2.$

19. $(x - 1)^2 = x + 2.$

20. $5x + x^2 + 6 = 0.$

21. $x^2 - 9x + 14 = 0.$

22. $x^2 + 3x - 70 = 0.$

23. $4x^2 - 4x - 3 = 0.$

24. $3x^2 - 7x + 2 = 0.$

25. $x^2 - 10x + 21 = 0.$

26. $x^2 - 10x + 24 = 0.$

27. $x^2 + 10x + 24 = 0.$

28. $x^2 = 9x^2 - (x + 1)^2.$

29. $9x^2 + 4x - 93 = 0.$

30. $4x^2 + 3x - 22 = 0.$

31. $6x^2 - 13x + 6 = 0.$

32. $(x + 1)(x + 2) = x + 3.$

33. $x + \frac{a^2}{x} = \frac{a^2}{b} + b.$

34. $cx^2 + bx + a = 0.$

35. $(x-2)(x+3) = 16.$

36. $\frac{a^2}{b+x} + \frac{a^2}{b-x} = c.$

37. $\frac{x + 1\frac{1}{3}}{x+2} - \frac{x+12}{2(x+19)} = \frac{1}{2}.$

38. $5x - \frac{3(x-1)}{x-3} = 2x + \frac{3(x-2)}{2}.$

39. $(x-1)^2(x+3) = x(x+5)(x-2).$

40. $x^2 - 6acx + a^2(9c^2 - 4b^2) = 0.$

41. $\frac{1}{a} + \frac{1}{b} + \frac{1}{x} - \frac{1}{a+b+x} = 0.$

42. $\frac{1}{a} + \frac{1}{a+x} + \frac{1}{a^2 - x^2} = 0.$

43. $(2a - 5 - x)^2 + (3a - 3x)^2 = (a + 5 - 2x)^2.$

44. $(3x - 4a + 3b)^2 + (2x + a)^2 = (x - 4a + b)^2 + (2x - 3a + 4b)^2.$

45. $(7a + 3b + x)^2 + (4a - b - 8x)^2 - (4a + 3b + 4x)^2 = (7a + b - 5x)^2.$

46. $(x + a)(5x - 3a - 4b) = (x + a - 2b)^2.$

47. $(5x + 4a + 3b)(10x - 6a + 8b) = (5x + a + 7b)^2.$

48. $(26x + a + 22b)(14x + 13a - 2b) = (16x + 11a + 8b)^2.$

49. $(8c + 10 + 4x)(18c + 160 + 24x) = (12c + 40 + 11x)^2.$

50. $\frac{14y^2 + 16}{21} - \frac{2y^2 + 8}{8y^2 - 11} = \frac{2y^2}{3}.$

51. $\frac{a^2 + x^2}{a^2} = \frac{(b+c)^2}{(b-c)^2}.$

52. $\frac{3x^2 - 27}{x^2 + 3} + \frac{90 + 4x^2}{x^2 + 9} = 7.$

53. $\frac{x^2 + x - 2}{x^2 + 2x - 3} - \frac{x+1}{x-3} = 0.$

54. $\frac{1}{a + b - x} = \frac{1}{a} + \frac{1}{b} - \frac{1}{x}.$

55. $(a - x)\left(1 - \frac{3a + 3x}{c - \frac{1}{2}}\right) - 2 = \frac{1 - 2a}{1 - 2c}(c - 3) - \frac{1 + a}{1 - \frac{1}{2c}}.$

56. $x^2 + xy + 4y^2 = 6,$
$3x^2 + 8y^2 = 14.$

57. $x^3 + y^3 = a,$
$x + y = b.$

58. $2x^2 + xy - 6y^2 = 3\frac{3}{4},$
$x + 2y = 2\frac{1}{2}.$

59. $x^2 + xy = 36,$
$x^2 - y^2 = -9.$

60. $x^2 + 3xy = 22,$
$x + y = 5.$

61. $x^2 + 4y^2 = 85,$
$x - y = 2.$

62. $x^2 - y^2 = 112,$
$x + y = 14.$

63. $x + y = 7,$
$x^3 + y^3 = 133.$

64. $x^3 + y^3 = 61,$
$x^2 - xy + y^2 = 61.$

65. $x^3 - y^3 = 665,$
$x - y = 5.$

66. $x^3 + y^3 = 28,$
$x^2y + xy^2 = 12.$

67. $x^2 - xy + y^2 = 37,$
$x + y = 10.$

68. $\dfrac{x}{x^2 + x + 5} + \dfrac{5}{\sqrt{x^2 + x + 5}} = \dfrac{24}{4x}.$

SUGGESTION. Multiply both members by x and let

$$y = \frac{x}{\sqrt{x^2 + x + 5}}.$$

69. $x^3 - 6x^{\frac{3}{2}} - 16 = 0.$

State what can be known by means of the discriminant and without solving, concerning the factors of:

70. $3x^2 - 2x + 1.$

71. $4x^2 + 11x - 1.$

72. $5y^2 + 20y + 20.$

73. $2x^2 - x + 3.$

Similarly, what can be known about the roots of:

74. $t^2 - 9 = 0$?

75. $5x^2 + x + 2 = 0$?

76. $z^2 - z - 1 = 0$?

77. $5x^2 = 9x$?

How must a be chosen in order that:

78. The roots of $x^2 + ax + 5 = 0$ shall be imaginary?

79. The roots of $ax^2 + 6x + 1 = 0$ shall be real?

80. The roots of $x^2 + 4x + 2a = 0$ shall be real and of opposite signs?

81. The roots of $(a + 1)x^2 + 3x - 2 = 0$ shall be imaginary?

82. The roots of $4x^2 - ax + 2 = 0$ shall be real and both positive?

Find the values of m for which the roots of the following equations are equal to each other. What are the corresponding values of x?

83. $x^2 - 12x + 3m = 0$.

84. $mx^2 + 8x + m = 0$.

85. $4x^2 + mx + x + 1 = 0$.

86. $mx^2 + 3mx - 5 = 0$.

87. A number increased by 30 is 12 less than its square. Find the number.

88. The product of two consecutive odd numbers is 99. What are the numbers? Is there more than one set?

89. Find two consecutive even numbers the sum of whose squares is 164.

90. Find a positive fraction such that its square added to the fraction itself makes $\frac{4}{9}$.

91. If a denotes the area of a rectangle and p its perimeter, show that the lengths of the sides are the roots of the equation

$$x^2 - \frac{p}{2}x + a = 0.$$

92. The diagonal and the longer side of a rectangle are together equal to 5 times the shorter side, and the longer side exceeds the shorter by 35 meters. Find the area of the rectangle.

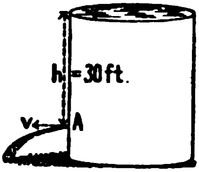
93. A company of soldiers attempts to form in a solid square, and 56 are left over. They attempt to form in a square with 3 more on each side, and there are 25 too few. How many soldiers are there?

94. It took a number of men as many days to dig a ditch as there were men. If there had been 6 more men, the work would have been done in 8 days. How many men were there?

95. Solve: $2x^2 + 6x + c = 0$.
What value must c have to make the two values of x equal?

96. A library spends \$180 monthly for books. In June the average cost per book was 15¢ less than in May, and 60 books more were bought. How many were bought in May?

97. When water flows from an orifice in a tank the square of the velocity (v) equals $2g$ times the height (h) of the surface above the orifice. Write the equation that denotes this fact. g is the "constant of gravity" and may be taken as 32.



98. What does the square of the velocity, (v^2), at A in the figure equal? Find this velocity.

99. What would be the velocity of the water if an opening were made halfway up from A shown in the figure?

100. Find the velocity with which water rushes through an opening at the base of a dam against which the water stands 25 ft. high.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. State the general forms of the roots of the general quadratic equation. Sec. 177.

2. State the relation between the *roots* and *coefficients*. Sec. 180.

3. Name two *symmetric* functions of the roots of the quadratic equation. Sec. 181.

4. What is the *discriminant* of the quadratic? Sec. 186.

5. What is the nature of the discriminant $b^2 - 4ac$ when the roots are:

   1. Real and unequal?
   2. Real and equal?
   3. Complex numbers?
   4. Irrational numbers? Sec. 185.

6. What methods are used to solve radical equations? Secs. 190–193.

7. What kind of higher equations may be solved like quadratics? Sec. 194.

8. What kind of equations does $x^n \pm a = 0$ represent? Sec. 195.

9. State the three cube roots of unity in terms of 1 and ω. Sec. 195.

10. What method of solving simultaneous quadratics has the broadest application? Sec. 197.

HISTORICAL NOTE

Among the many improvements in algebra made in the sixteenth century was that masterly achievement, the general solution of the cubic equation. We have explained how for centuries the solution of the quadratic equation $x^2 + px = q$ baffled the skill of the keenest mathematicians, but the solution of the cubic $x^3 + px^2 = q$ had been regarded quite unattainable, so much so that the great Italian Paciuolo (1494) closed his famous work *Summa* with the remark that the solution of this equation was as impossible as the squaring of the circle. But to assert that the solution of a problem is impossible is dangerous ground to take, as was proved in this case. For within fifty years after this remark of Paciuolo, his countryman Niccolo solved the cubic equation.

NICCOLO, OR TARTAGLIA

Niccolo was born at Brescia just at the beginning of the sixteenth century, and came to be known as Tartaglia, meaning, "The stammerer." This defect in his speech was caused by a saber cut received at the hands of a French soldier when Niccolo was six years old. Although poverty deprived him of the advantages of school instruction, his industry and persistence enabled him to master the classics and mathematics, and to add (1541) to the science of algebra the most significant discovery of the sixteenth century. It was Tartaglia's ambition to write a great work on algebra and to use this as a means of giving to the world his method of solving the cubic equation. But before he could accomplish this, he was betrayed by his friend Cardan, to whom, after much pleading, he had revealed the nature of his discovery. Cardan published his now famous work, the *Ars Magna*, in 1545, and included, as the crown to his treatment of equations, the solution by Tartaglia, claiming for himself full credit for the discovery. This deception and robbery was a bitter disappointment to Tartaglia, who never recovered sufficiently to complete his projected writings, and, although the claim of Cardan is now conceded by all to be fraudulent, the solution of the cubic equation is still commonly called Cardan's method.

The solution of the equation of the fourth degree immediately followed that of the cubic equation. This was first effected, in a similar way, by Ferrari, a pupil of Cardan, and was also published in the *Ars Magna.* It was naturally supposed that these methods could be extended to equations of degrees higher than the fourth, and prodigious labor was expended in the effort to do this, until Abel, a Norwegian mathematician, proved in 1824 that the methods of elementary algebra are not sufficient to solve general equations of degree higher than the fourth.

CHAPTER VIII

PROPORTION, VARIATION, AND LIMITS

PROPORTION

198. Proportion. If four numbers, a, b, c, d, are such that $\frac{a}{b}=\frac{c}{d}$, they are said to form the proportion, ***a* is to *b* as *c* is to *d*.**

199. Fourth Proportional. If four numbers a, b, c, d are in proportion, d is called the *fourth proportional* to a, b, and c.

200. Third Proportional. If three numbers, a, b, c, are such that a, b, b, c are in proportion, c is called the *third proportional* to a and b.

201. Mean Proportional. If a, b, b, c are in proportion, b is called the *mean proportional* to a and c.

202. Means and Extremes. If a, b, c, d are in proportion, b and c are called the *means*, and a and d the *extremes*.

203. Relation of Means to Extremes. *In any proportion the product of the means equals the product of the extremes, and conversely, if the product of two numbers equals the product of two other numbers, the four numbers can be arranged in a proportion.*

For, the two factors of one product may be made the means, and the two factors of the other the extremes, or *vice versa*.

204. Inversion. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

For, the members of the second equation are the reciprocals of the members of the first.

205. Alternation. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

For, the second equation results if the first is multiplied by $\frac{b}{c}$.

206. Composition. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

For, the second equation results from the first by adding 1 to both members.

207. Division. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

For, the second equation results from the first if 1 be subtracted from both members.

208. Composition and Division. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

For, the result of Sec. 206 divided by the result of Sec. 207 gives the second equation.

209. Continued Proportion. If several ratios are equal, as in $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}, \ldots$ the numbers, a, b, c, d, e, $f, \ldots$, are said to be in continued proportion.

In the continued proportion $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}, \ldots$, any one of the ratios equals $\frac{a+c+e\ldots}{b+d+f\ldots}$.

210. These properties may be applied to certain problems.

EXAMPLES

1. If p and w are the power and weight applied to a lever whose power and weight arms are, respectively, a and b, then $pa = bw$. Write a proportion between these four numbers.

SOLUTION. By Sec. 203, $a : b = w : p$.

2. Find the third proportional to $a - b$ and $a^2 - b^2$.

SOLUTION. 1. By Sec. 203, $\frac{a-b}{a^2-b^2}=\frac{a^2-b^2}{x}$.

2. $\therefore (a-b)x = (a^2-b^2)(a^2-b^2)$.

3. $\therefore$ the third proportional, $x = (a+b)(a^2-b^2)$.

3. Form a proportion from the equation, $x^2 - y^2 = z^2$.

SOLUTION. 1. Factoring, $(x-y)(x+y) = zz$.

2. $\therefore \frac{x-y}{z}=\frac{z}{x+y}$. Sec. 203.

4. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{c+d}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$. (*1*)

By Sec. 206 $\qquad \frac{a+b}{b}=\frac{c+d}{d}.$ (*2*)

∴ by Sec. 205, $\qquad \frac{a+b}{c+d}=\frac{b}{d}.$ (*3*)

Squaring both members of the given equation, $\qquad \frac{a^2}{b^2}=\frac{c^2}{d^2}.$ (*4*)

Applying Sec. 206 to (*4*), $\qquad \frac{a^2+b^2}{b^2}=\frac{c^2+d^2}{d^2}.$ (*5*)

Applying Sec. 205 to (*5*), $\qquad \frac{a^2+b^2}{c^2+d^2}=\frac{b^2}{d^2}.$ (*6*)

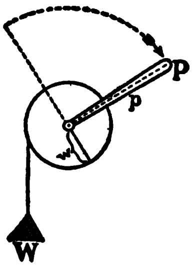
From (*6*), $\qquad \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{b}{d}.$ (*7*)

Equating values of $\frac{b}{d}$ in (*3*) and (*6*), $\qquad \frac{a+b}{c+d}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}.$ (*8*)

WRITTEN EXERCISES

1. A lever need not be straight, although it must be rigid. Thus, the crank and the wheel and axle are varieties of the lever, and the law of the lever (p. 188, Part I) applies to them. Thus, in the figure,

$$\frac{P}{W}=\frac{w}{p}.$$

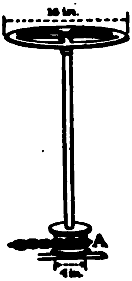


Find W, if $P=14, p=16$, and $w=4$.

2. Find the unknown number:

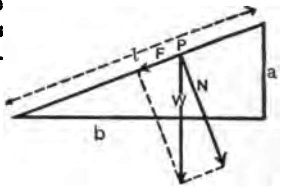
	(1)	(2)	(3)
$P=$	$3a$	—	$a-b$
$W=$	$6b$	$5p$	$(a+b)^2$
$p=$	$2c$	$8p$	$a+b$
$w=$	—	$2p$	—

3. If an axle is 6 in. in diameter, what must be the diameter of the wheel in order that a boy exerting a force of 50 lb. may be able to raise 800 lb. weight?



4. A brakeman pulls with a force of 150 lb. on a brake wheel 16 in. in diameter. The force is communicated to the brake by means of an axle, A, 4 in. in diameter. What is the pull on the brake chain?

5. In the figure below the weight W acts at P on an inclined plane, whose rate of slope is a vertical units to b horizontal units. It is known that the weight W acts in two ways: a force N pressing directly against the surface and tending to produce friction, and a force F parallel to the plane and tending to cause the weight to slide down the plane. It is known that these various quantities are related to each other thus:



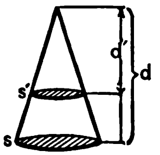
$$\frac{F}{W}=\frac{a}{l}. \qquad \frac{N}{W}=\frac{b}{l}.$$

Find F and N, if $W = 15$ lb., $a = 20$ in., $b = 21$ in., $l = 29$ in.

6. Find l and N, if $F = 16$ lb., $W = 34$ lb., $a = 4$ ft. $b = 7\frac{1}{2}$ ft.

7. Find b and l, if $F = 66$ lb., $N = 112$ lb., $W = 130$ lb., $a = 33$ in.

8. Find W and a, if $F = 200$ lb., $N = 45$ lb., $b = 9$, $l = 41$.



9. It is known that in any cone the area of any section parallel to the base varies as the square of its distance from the vertex. Thus, in the figure, $\frac{s'}{s}=\frac{d'^2}{d^2}$.

If $d = 1$, $d' = \frac{1}{2}$, and $s = 40$ sq. in., what is the area of s'?

10. The area of a section $\frac{1}{3}$ of the way from the vertex to the base and parallel to it is what part of the base?

11. Given $\frac{a}{b}=\frac{b}{c}$, show that $\frac{a^2+b^2}{a+c}=\frac{a^2-b^2}{a-c}$.

12. Given $\frac{a}{b}=\frac{b}{c}$, show that $\frac{a^2-b^2}{a}=\frac{b^2-c^2}{c}$.

13. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a}{d(a+b)^2}=\frac{c}{b(d+c)^2}$.

14. If $a:b=p:q$, prove that

$$a^2+b^2:\frac{a^3}{a+b}=p^2+q^2:\frac{p^3}{p+q}.$$

Suggestion. The given proportion may be written:

$$\frac{a}{p}=\frac{b}{q}.$$

Let $\frac{a}{p}=r$, then $a=pr$, $b=qr$.

The proportion to be proved may be written:

$$\frac{(a+b)(a^2+b^2)}{a^3}=\frac{(p+q)(p^2+q^2)}{p^3}.$$

Substituting the values of a and b above, the left member readily reduces to the right.

Note. A good method for proving such identities is to begin with the required relation and transform it into the given relation, or to transform both the given and the required relation until they reduce to the same thing.

15. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{4a^2-5b^2}{5b^2}=\frac{4c^2-5d^2}{5d^2}$.

16. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a^2-b^2}{c^2-d^2}=\frac{b^2}{d^2}$.

17. Write $x^2-4y^2=x^2-xy$ in the form of a proportion.

18. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a}{c}=\frac{\sqrt{a^2-b^2}}{\sqrt{c^2-d^2}}$.

VARIATION

211. Direct Variation. When two variable quantities vary so as always to remain in the same ratio, each is said to **vary directly** as the other. Each increases or decreases at the same rate that the other increases or decreases.

Consequently, if x and y are two corresponding values of the variables and k the fixed ratio, then,

$$\frac{x}{y} = k \text{ and } x = ky.$$

For example, at a fixed price (k) per article, the total cost (x) of a number of articles of the same sort varies directly with the number of articles (y). That is $\frac{x}{y} = k$.

Likewise in the case of motion at a uniform rate (r), the distance traversed (d) varies as the time of motion (t). That is, $\frac{d}{t} = r$.

212. A symbol still occasionally used for "varies as" is $\propto$.

Thus, "x varies as y" is written $x \propto y$,
and "d varies as t" is written $d \propto t$.

213. Relation of Variation to Proportion. When one variable varies directly as another, any pair of values of the variables forms a proportion with any other pair.

For, $\frac{x'}{y'} = r$, and $\frac{x''}{y''} = r$; $\therefore \frac{x'}{y'} = \frac{x''}{y''}$, which is a proportion.

214. Expressions for Direct Variation. We have thus seen that the relation *x varies directly as y* may be expressed in any one of three ways:

(*a*) $x = ky$, by use of the equation.

(*b*) $x \propto y$, by use of the symbol of variation.

(*c*) $\frac{x'}{y'} = \frac{x''}{y''}$ by use of the proportion; x', y' and x'', y'' being any two pairs of corresponding values of the variables.

WRITTEN EXERCISES

1. Write the statement "v varies as w" in the form of an equation, also in the form of a proportion.

2. Write the statement $x = ky$ by use of the symbol $\propto$; also in the form of a proportion.

3. Write $\frac{t'}{v'} = \frac{t''}{v''}$ by use of the symbol $\propto$.

4. The weight (w) of a substance varies as the volume (v) when other conditions are unchanged. Express this law by use of the equation. By use of the symbol $\propto$. Also in the form of a proportion.

5. In the equation $w = kv$, if $w = 4$ and $v = 2$, what is the value of k? Using this value of k, what is the value of w when $v = 25$?

6. When 1728 cu. in. of a substance weigh 1000 ounces, what is the ratio of the weight (w) to the volume (v)? What volume of this substance will weigh 5250 ounces?

7. The cost (c) of a grade of silk varies as the number of yards (n). Find the ratio (r) of c to n when c is \$7.00 and $n = 4$.

8. In Exercise 7 if $\frac{c}{n} = r$, what does $\frac{c'}{n'}$ equal? Given that 40 yd. of silk cost \$60, find the cost of 95 yd. by means of the proportion $\frac{c'}{n'} = \frac{c}{n}$. Also by means of the equation $c = nr$.

215. Inverse Variation. A variable x is said to **vary inversely** as a variable y, if it varies directly as $\frac{1}{y}$.

Inverse variation means that when one variable is doubled the other is halved; when one is trebled the other becomes $\frac{1}{3}$ of its original value, and so on.

216. Expressions for Inverse Variation. The relation "x varies inversely as y" may be expressed:

(*a*) By the equation, $x = k\left(\frac{1}{y}\right)$, or $x = \frac{k}{y}$. $\therefore xy = k$.

(*b*) With the symbol of variation $y \propto \frac{1}{x}$.

(*c*) As a proportion $\frac{x'}{x''} = \frac{y''}{y'}$.

WRITTEN EXERCISES

1. Write the statement "v varies inversely as w" in the form of an equation; also in the form of a proportion.

2. Write $t = \frac{k}{p}$ by use of the symbol $\propto$; also in the form of a proportion.

3. In a bicycle pump the volume (v) of air confined varies inversely as the pressure (p) on the piston. Write the relation between v and p in three ways.

4. In Exercise 3, if $v = 18$ (cu. in.) and $p = 15$ (lb.), what is k in $v = \frac{k}{p}$? What is the pressure (p) when $v = 1$ (cu. in.)?

5. In an auditorium whose volume (v) is 25,000 cu. ft. there are 2000 persons (p). What is the number (n) of cubic feet of air space to the person? What will be the number when 1000 more persons come in?

6. The area of a triangle varies as the base times the altitude. If the area is 12 when the base is 8 and the altitude 3, what is the area of a triangle whose base is 40 and altitude 20?

7. The area of a circle varies as the square of its radius. The area of a circle of radius 2 is 12.5664; what is the area of a circle whose radius is 5.5?

8. The volume of a sphere varies as the cube of its radius. If the volume of a sphere whose radius is 3 is 113.0976, what is the volume of a sphere whose radius is 5?

9. If $x \propto y$ and $x = 6$ when $y = 2$, find x when $y = 8$.

SUGGESTION. $x = ky$. $\therefore 6 = k \cdot 2$ and $k = 3$. Substitute $y = 8$ in the equation $x = 3y$.

10. Determine k in $x \propto y$, if $x = 10$ when $y = 20$. Also if $x = 1$ when $y = 5$. If $x = 100$ when $y = 10$.

11. If $x \propto w$ and $y \propto w$, prove that $x + y \propto w$.

12. If $x \propto w$ and $w \propto y$, prove that $xy \propto w^2$.

13. If $x \propto y$ and $w \propto z$, prove that $\frac{x}{w} \propto \frac{y}{z}$.

14. If $x \propto y^2z$ and $x = 1$ when $y = 2$ and $z = 3$, find the constant k.

15. Given $y = z + w$, $z \propto x$ and $w \propto x$; and that $x = 1$ when $w = 6$, and that $x = 2$ when $z = 20$. Express y in terms of x.

Suggestion. $y = kx + k'x$; determine k and k'.

16. Solve Exercise 15 under the conditions that $x = 2$ when $w = 12$, and that $x = 1$ when $z = 10$.

17. Given $z \propto x + y$ and $y \propto x^2$, and that $x = \frac{1}{2}$ when $y = \frac{1}{8}$ and $z = \frac{1}{4}$. Express z in terms of x.

217. A large number of problems in science may be solved by the following plan:

EXAMPLES

1. The "law of gravitation" states that the weight of a given body varies inversely as the square of its distance from the center of the earth. What is the weight of a body 5 mi. above the surface of the earth, which weighs 10 lb. at the surface (4000 mi. from the center)?

Method. There are two variables in the problem, the weight (w) and the distance (d). There are also two parts or cases in the statement, one in which the value of one variable is unknown, and one in which the values of both variables are given.

Arrange the data as follows:

	w	d
1st case	x	4005
2d case	10	4000

To this table apply the law of variation expressed in the physical law. Since the law is: *w varies inversely as the square of d*, the values of d must be squared, and the ratio $x : 10$ equals the inverse ratio of 4005^2 to 4000^2; that is,

$$\frac{x}{10} = \frac{4000^2}{4005^2}.$$

$$4000 \div 4005 = .99874 + \text{ and } .9987^2 = 9.96 \ldots$$

$$\therefore x = 9.96, \text{ and the weight is } 9.96 \text{ lb.}$$

2. The squares of the times of revolution of the planets about the sun vary directly as the cubes of their distances from the sun. The earth is 93,000,000 mi. from the sun, and makes a revolution in approximately 365 da. How far is Venus from the sun, its time of revolution being 226 da.?

Solution

	t = time of revol.	d = distance from sun
1st case	365	93,000,000
2d case	226	x

According to the astronomical law the times must be squared and the distances cubed; then, since the law is that of direct variation:

$$\frac{365^2}{226^2}=\frac{93,000,000^3}{x^3}.$$

$\therefore x = 93,000,000\sqrt[3]{\left(\frac{226}{365}\right)^2} = 68,900,000$, and the distance of Venus from the sun is approximately 69,000,000 miles.

WRITTEN EXERCISES

1. The intensity of light from a given source varies inversely as the square of the distance from the source. If the intensity (candle power) of an electric light is 4 at a distance of 150 yd., what is its intensity at a distance of 25 yd.?

2. According to the first sentence of Exercise 1, how much farther from an electric light must a surface be moved to receive only $\frac{1}{4}$ as much light as formerly?

3. The time of oscillation of a pendulum varies directly as the square root of its length. What is the length of a pendulum which makes an oscillation in 5 sec., a 2-second pendulum being 156.8 in. long?

4. According to Exercise 3, what is the time of oscillation of a pendulum 784 in. long?

5. The distance through which a body falls from rest varies as the square of the time of falling. A body falls from rest 576 ft. in 6 sec.; how far does it fall in 10 sec.?

6. Volumes of similar solids vary as the cubes of their linear dimensions. The volume of a sphere of radius 1 in. is 4.1888 cu. in.; what is the volume of a sphere whose radius is 5 in.?

7. According to Exercise 6, what is the radius of a sphere whose volume is 33.5104 cu. ft.?

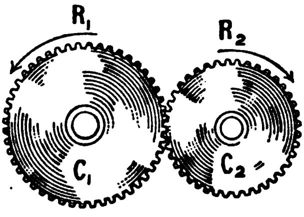
8. According to Exercise 6, if a flask holds $\frac{1}{2}$ pt., what is the capacity of a flask of the same shape 4 times as high?

9. In compressing a gas into a closed receptacle, as in pumping air into an automobile tire, the pressure varies inversely as the volume. If the pressure is 25 lb. when the volume is 125 cu. in., what is the pressure when the volume is 115 cu. in.?

10. According to Exercise 9, if the pressure is 50 lb. when the volume is 250 cu. in., what is the volume when the pressure is 10 lb.?

11. It is known that if one gear wheel turns another as in the figure, the number of revolutions of the two are to each other inversely as their number of teeth. That is, if the first has C_1 teeth and makes R_1 revolutions, and the second has C_2 teeth and makes in the same time R_2 revolutions,

then $$\frac{R_1}{R_2} = \frac{C_2}{C_1}.$$



Find R_2 if $C_1 = 25$, $C_2 = 15$, and $R_1 = 6$.

Find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)
$C_1 =$	42	60	$5n$	—
$C_2 =$	—	50	$3n$	40
$R_1 =$	12	—	21	$8n$
$R_2 =$	9	15	—	$6n$

12. A diamond worth $2000 was broken into two parts, together worth only $1600. If the value of a diamond is proportional to the square of its weight, into what fractions was the original diamond broken? (Find result to nearest hundredth.)

LIMITS

218. In the construction of graphs we have studied the changes in functions corresponding to given values of their variables, but we shall now consider an important particular case in which the successive values of a variable approach nearer and nearer to a fixed number.

For example, if we take the values of the decimal .444 . . . corresponding to the successive decimal places, we have .4, .44, .444, .4444, . . . , a series of increasing numbers so limited that no one of them, however far we go, can be as large as .45. In fact, no one of them can equal $\frac{4}{9}$, but by using more and more decimal places we may come to terms differing as little from $\frac{4}{9}$ as we choose.

219. Limit. If a variable x has a boundless number of successive values which approach nearer and nearer to a fixed number l, so that the difference $l - x$ may become and remain numerically as small as we choose, *x is said to approach l as a limit.*

220. This relation is expressed in symbols by $x \doteq l$, read, "x approaches l as a limit."

For example: In Sec. 218, if the variable x be taken to represent the different numbers in the series .4, .44, .444, . . . , then $x \doteq \frac{4}{9}$.

Or, if x is successively .02, .002, .0002, and so on as far as we choose, then $x \doteq 0$.

221. Meaning of $\frac{0}{0}$. We have shown (Sec. 101, II) how the indicated operation, $\frac{0}{0}$, may arise from solving systems of equations, and that its value is indeterminate. We will further illustrate it here by use of expressions having limiting values.

For example, the fraction $\frac{x-1}{x^2-1}$ becomes $\frac{0}{0}$, if $x = 1$, but it can be shown to have the limiting value $\frac{1}{2}$. If we let $x = .9$, .99, .999, then $\frac{x-1}{x^2-1}$ becomes $\frac{10}{19}$, $\frac{100}{199}$, $\frac{1000}{1999}$, a series of fractions whose values decrease toward $\frac{1}{2}$ when $x \doteq 1$. Furthermore, $\frac{x-1}{x^2-1} = \frac{1}{x+1}$ by reducing to lowest terms, and $\frac{1}{x+1} \doteq \frac{1}{2}$, when $x \doteq 1$. Thus, $\frac{x-1}{x^2-1}$ has the limiting value $\frac{1}{2}$ when $x \doteq 1$.

Similarly, the fraction $\frac{x}{x^2 - x}$ has the limiting value -1, when $x \doteq 0$.

Thus the expression $\frac{0}{0}$ may represent different numbers, and as a symbol taken by itself, it must be regarded as indeterminate.

222. Meaning of ∞. Opposed to quantities which tend toward zero, are quantities which grow large without bound. Such variables are said to have the property of becoming infinitely great.

For example, if x assumes in turn all integral values, 1, 2, 3, 4, and so on, without end, x is said to have the property of becoming infinite.

The symbol, ∞, called *infinity*, is commonly used to express the fact that a variable has the above property, but it must not be regarded as a particular *number* nor as a *limit;* it is merely a sign of the infinitely great.

Thus, $n = \infty$ means "when n becomes infinitely great."

223. Meaning of $\frac{a}{\infty}$. This means a fixed number divided by a variable number which grows large without bound. Under these conditions, $\frac{a}{\infty} \doteq 0$.

For example, let a be represented by a line 1 ft. long, and suppose it to be bisected. The result is $\frac{12 \text{ in.}}{2} = 6$ in. Then, suppose each division to be bisected again. The result is $\frac{12 \text{ in.}}{4} = 3$ in. Bisect each division again, and suppose the process to be continued indefinitely. The denominator becomes 32,768 when the division has been made 15 times, and $\frac{12 \text{ in.}}{32{,}768}$ is less than $\frac{1}{2730}$ in. By taking more and more divisions, the fraction expressing the length of a division approaches zero as a limit.

224. Meaning of $\frac{\infty}{\infty}$. This means the quotient of two variable numbers each of which grows large without bound, and the expression can have no fixed value. But the quotient may tend to a limit as both numerator and denominator increase.

For example, when $x = \infty$, $\frac{x}{1 + x} = \frac{\infty}{\infty}$, but it has the limiting value 1, as may be seen by dividing both terms by x; thus, $\frac{x}{1 + x} = \frac{1}{1 + \frac{1}{x}} \doteq \frac{1}{1 + 0}$ or 1, when $x = \infty$.

WRITTEN EXERCISES

What is the limiting value of each of the following when $n=\infty$:

1. $\frac{1}{x}$.

2. $\frac{3}{x}$.

3. $\frac{5}{y}$.

4. $\frac{y}{y-2}$.

5. $\frac{z+1}{z}$

6. $\frac{z-5}{z}$.

7. $\frac{x(x+2)}{x^2}$. (Reduce and separate into a whole number and a fraction before substituting $x=\infty$.)

8. Apply the suggestion in Ex. 7 to $\frac{x(x-1)(x-2)}{x^3}$.

What value does each expression approach as $v \doteq 0$:

9. $\frac{1}{v}$?

10. $\frac{\frac{3}{2}}{v}$?

11. $3v^2$?

12. $\frac{v}{v+1}$?

Find the limiting value of:

13. $\frac{z-2}{z^2-4}$ as $z \doteq 2$.

14. $\frac{z^2-9}{z-3}$ as $z \doteq 3$.

15. $\frac{x^2+2x-3}{x^2-1}$ as $x \doteq 1$.

16. $\frac{x^2-4}{x^2+x-2}$ as $x \doteq -2$.

GRAPHICAL WORK

225. The relation "x varies as y" has been expressed by the linear equation $x=ky$, and is represented by a *straight line* (Sec. 287, p. 205, Part I).

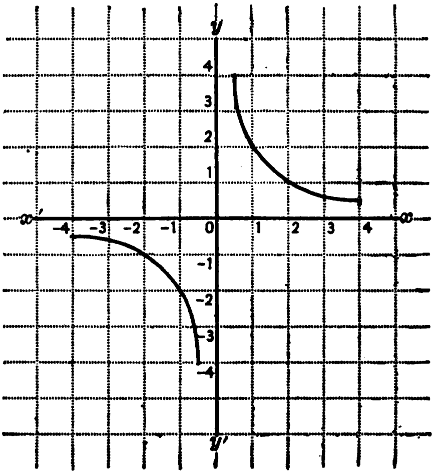
The relation "x varies inversely as y" has been expressed by the equation $x=\frac{k}{y}$. The graph of this equation is a *curve*.

Suppose, for illustration, that $k = 2$. Then $x = \frac{2}{y}$. For this equation:

The table is

x	y
5	$\frac{2}{5}$
4	$\frac{1}{2}$
3	$\frac{2}{3}$
2	1
1	2
$\frac{1}{2}$	4
-1	-2
-2	-1
-3	$-\frac{2}{3}$
-4	$-\frac{1}{2}$
-5	$-\frac{2}{5}$

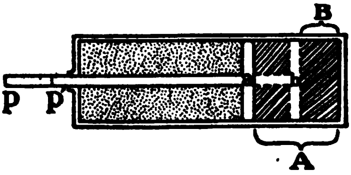
The graph is



If large numerical values are given to y, x will become correspondingly small so that for $y = \infty$, $x \doteq 0$. Thus, the curves approach nearer and nearer to the y-axis. Similarly, when $x = \infty$, then $y \doteq 0$, and the curves approach nearer and nearer to the x-axis.

WRITTEN EXERCISES

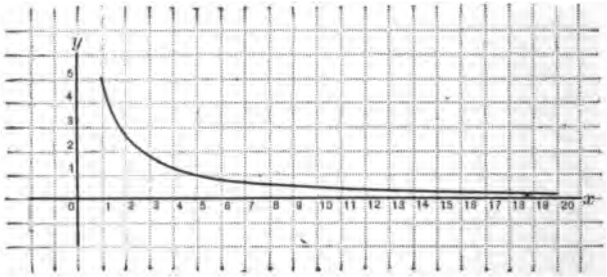
1. Under standard conditions the volume (v) of a confined gas varies inversely as the pressure (p). That is, $v = \frac{k}{p}$. This is known as Boyle's Law. Suppose when the piston is at a in the figure, the volume of the gas in part A is 1 cu. ft., and the pressure at P is 5 lb. When the pressure is 10 lb. the piston moves to b, and the volume of the gas B becomes $\frac{1}{2}$ cu. ft. What will be the volume of the gas when $P = 20$ lb.?



Taking $k = 5$ in the equation $v = \frac{k}{p}$, the table of values is

$p =$	1	2	3	4	5	6	7	8	9	10	15	20
$v =$	5	$2\frac{1}{2}$	$1\frac{2}{3}$	$1\frac{1}{4}$	1	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

The graph of the table is:



Read from the graph the pressure on the piston necessary to hold this volume of gas at $\frac{1}{2}$ cu. ft.; at $\frac{1}{4}$ cu ft.; at 5 cu. ft.; at 2 cu. ft.

What is the limiting value of v when $p = \infty$?

Represent graphically:

2. $x = \frac{1}{y}$, or $xy = 1$.

4. $x = \frac{4}{y}$, or $xy = 4$.

3. $x = \frac{-2}{y}$, or $xy = -2$.

5. $3x = \frac{-1}{y}$, or $3xy = -1$.

6. The attraction or "pull" of the earth on bodies in its neighborhood is the cause of their weight. The law of gravitation states that the weight of a given body varies inversely as the square of its distance from the center of the earth. This law may be expressed by the equation $w = \frac{k}{d^2}$.

To construct the graph which shows the nature of the relation between w and d as they vary, k may be taken to be 1.

Fill the blanks in the table:

d	± 1	± 5	$\pm \frac{1}{3}$	$\pm .25$	$\pm .2$
w	1	( )	( )	( )	( )

7. Plot the graph for the table in Exercise 6.

REVIEW

WRITTEN EXERCISES

1. Solve, using the principle of composition and division,

$$(a-\sqrt{2bx+x^2}):(a-b)=(a+\sqrt{2bx+x^2}):(a+b).$$

2. If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}$, prove that

$$\frac{a_1+a_2+a_3+\cdots+a_n}{b_1+b_2+b_3+\cdots+b_n}=\frac{a_1}{b_1}.$$

3. Given $x+\sqrt{x}:x-\sqrt{x}::3\sqrt{x}+6:2\sqrt{x}$, to find x.

4. If the volume of a sphere varies as the cube of its radius, find the radius of a sphere whose volume equals that of the sum of two spheres whose radii are, respectively, 6 ft. and 3.5 ft.

5. The number of vibrations (swings) made by two pendulums in the same time are to each other inversely as the square roots of their lengths. If a pendulum of length 39 in. makes 1 vibration per second (called a seconds pendulum), about how many vibrations will a pendulum 10 in. long make? How long must the pendulum be to make 10 vibrations per second?

6. Two towns join in building a bridge which both will use, and agree to share its cost, $5000, in direct proportion to their populations and in inverse proportion to their distances from the bridge. One town has a population of 5000 and is 2 mi. from the bridge; the other has a population of 9000 and is 6 mi. from the bridge. What must each pay?

7. In $x=\frac{3}{y}$ what is the limiting value of x when $y=\infty$?

8. In $x=\frac{a}{\frac{1}{y}}$ what does x become when $y=\infty$?

9. Find the limiting value of $\frac{x^2-25}{x^2+2x-15}$ as $x \doteq -5$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define a *proportion;* define *means;* also *extremes.* Secs. 198, 202.

2. What is a *fourth proportional?* A *third proportional?* A *mean proportional?* Secs. 199–201.

3. Define *alternation;* also *inversion; composition; division.* Secs. 204–208.

4. What is a *continued proportion?* Sec. 209.

5. Define and illustrate *direct variation.* Sec. 211.

6. State the relation of variation to proportion. Sec. 213.

7. Define and illustrate *inverse variation.* Sec. 215.

8. State an equation expressing the law of direct variation; also one expressing inverse variation. Secs. 214, 216.

9. Define and illustrate *limit.* Sec. 219.

10. Explain the meaning of the symbol $\frac{0}{0}$; also ∞; also $\frac{a}{\infty}$; also $\frac{\infty}{\infty}$. Secs. 221–224.

11. What kind of a line is the graph of the equation expressing direct variation? Of the equation expressing inverse variation? Sec. 225.

CHAPTER IX

SERIES

226. Series. If a sequence of numbers is determined by a given law, the sequence of numbers is called a **series.**

227. Terms. The numbers constituting the series are called its **terms,** and are named from the left, 1st term, 2d term, etc.

The following are examples of series:

1. $1, 2, 3, 4, 5, \cdots.$
2. $1, 3, 5, 7, 9, \cdots.$
3. $1, 5, 9, 13, 17, \cdots.$
4. $3, 6, 9, 12, 15, \cdots.$
5. $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \cdots.$
6. $2, 4, 8, 16, 32, \cdots.$
7. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots.$
8. $1, 3, 9, 27, 81, 243, \cdots.$
9. $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \cdots.$
10. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots.$
11. $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \cdots.$
12. $100, 99, 98, 97, 96, 95, \cdots.$

ORAL EXERCISES

1–12. State the next five terms of each series above.

228. When the *law* of a series is known, any term may be found directly.

EXAMPLES

1. The law of the second series in Sec. 227 is that each term is two more than the preceding. To get the tenth term we start from 1 and add 2 for each of 9 terms.

That is, the tenth term is $1 + 9 \cdot 2 = 19.$

Similarly, the 12th term is $1 + 11 \cdot 2 = 23,$
the 15th term is $1 + 14 \cdot 2 = 29,$
the 47th term is $1 + 46 \cdot 2 = 93,$
the nth term is $1 + (n - 1)2 = 2n - 1.$

2. The law of the eighth series is that each term after the first is 3 times the preceding term. To get the ninth term we start at 1 and multiply by 3 eight times, or by 3^8. That is, the ninth term is $1 \cdot 3^8 = 6561$.

Similarly, the 12th term is $1 \cdot 3^{11} = 177{,}147$,
the nth term is $1 \cdot 3^{n-1} = 3^{n-1}$.

WRITTEN EXERCISES

1–4. Select four of the series in Sec. 227 that can be treated like the first example and write the 10th, 12th, 15th, and 47th terms of each.

5–7. Select three of the series in Sec. 227 that can be treated like the second example and write the 8th, 10th, and nth terms of each.

8. Write similarly the 7th, 11th, 20th, 47th, and nth term for any 6 of the above series.

229. We shall give only two types of series, the arithmetical and the geometric, the laws of which are comparatively simple.

ARITHMETICAL SERIES

230. Arithmetical Series. A series in which each term after the first is formed by adding a fixed number to the preceding term is called an **arithmetical series** or **arithmetical progression.**

231. Common Difference. The fixed number is called the **common difference**, and may, of course, be negative.

For example:

1. 7, 15, 23, 31, 39, ⋯ is an arithmetical series having the common difference 8.

2. 16, $14\frac{1}{2}$, 13, $11\frac{1}{2}$, 10, ⋯ is an arithmetical series having the common difference $-\frac{3}{2}$.

WRITTEN EXERCISES

1. Select the arithmetical series in the list of Sec. 227. Write the nth term in each.

2. Beginning with 2 find the 100th even number.

3. Beginning with 1 find the 100th odd number.

4. Beginning with 3 find the 200th multiple of 3.

5. A city with a population of 15,000 increased 600 persons per year for 10 yr. What was the population at the end of 10 yr.?

232. A general form for an arithmetical series is:

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d,\ \cdots,$$

where a denotes the first term,
d denotes the common difference, and
n denotes the number of the term.

233. Last Term. If the last term considered is numbered n and denoted by l, we have for the last of n terms the formula:

$$l=a+(n-1)d.$$

234. The Sum of an Arithmetical Series. The sum of n terms of an arithmetical series can be found readily.

EXAMPLE

Find the sum of the first 6 even numbers.

1. Let $s=2+4+6+8+10+12.$
2. We may also write $s=12+10+8+6+4+2.$
3. Adding (1) and (2),

$$2s=(2+12)+(4+10)+(6+8)+(8+6)+(10+4)+(12+2)$$
$=6(2+12)$; for each parenthesis is the same as $2+12$.

4. $\therefore s=\frac{6(2+12)}{2}=42.$

WRITTEN EXERCISES

Find similarly the sum of:

1. The first 6 odd numbers.
2. The first 6 multiples of 3.
3. The first 5 multiples of 7.
4. The first 4 multiples of 8.

235. General Formula for the Sum. The general form of the series may be treated in the same way.

If l denotes the last of n terms, the term before it is denoted by $l-d$, the next preceding by $l-2d$, and so on. Hence, the sum of n terms may be written:

$$s=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l.$$

And also, $s=l+(l-d)+(l-2d)+\cdots+(a+2d)+(a+d)+a.$

Whence, adding, $2s=(a+l)+(a+l)+\cdots+(a+l)=n(a+l).$

Therefore, $$s=\frac{n(a+l)}{2}.$$

Or, in words,

The sum of any number of terms of an arithmetical series is one half the sum of the first and the last terms times the number of terms.

By using the value of l in Sec. 233, $s=\frac{n[2a+(n-1)d]}{2}.$

This permits the calculation of s without working out separately the value of l.

WRITTEN EXERCISES

For each series in the following list find: First, the sum of 10 terms. Second, the sum of n terms.

1. 1, 2, 3, 4, 5, ⋯.
2. 1, 3, 5, 7, 9, ⋯.
3. 1, 5, 9, 13, 17, ⋯.
4. 3, 6, 9, 12, 15, ⋯.
5. 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, ⋯.
6. 100, 99, 98, 97, 96, 95, ⋯.

7. A man invests \$100 of his earnings at the beginning of each year for 10 yr. at 6 %, simple interest. How much has he at the end of 10 yr.?

Solution.

1. The last investment bears interest 1 yr. and amounts to \$106; the next to the last bears interest 2 yr. and amounts to \$112, etc.; the first bears interest 10 yr. and amounts to \$160.
2. Hence, $a=\$106$, $d=\$6$, and $n=10$.
3. Therefore, $l=106+9\cdot 6=160$.
4. Therefore, $s=\frac{10}{2}(106+160)=1330$.
5. The man has \$1330.

8. If \$50 is invested at the beginning of each year for 20 yr. at 5 % simple interest, what is the amount at the end of 20 yr.?

9. If a body falls approximately 16 ft. the first second and 32 ft. farther in each succeeding second, how far does it fall in 5 sec.?

236. Collected Results. The three chief formulas of arithmetical series are:

$$1.\quad l = a + (n-1)d.$$

$$2.\quad s = \frac{n(a+l)}{2}.$$

$$3.\quad s = \frac{n[2a+(n-1)d]}{2}.$$

GEOMETRIC SERIES

237. Geometric Series. A series in which each term after the first is formed by multiplying the preceding term by a fixed number is called a **geometric series,** or a **geometric progression.**

238. Common Ratio. The fixed multiplier is called the **common ratio,** and may be negative.

For example:

1. 2 is the common ratio in the geometric series 2, 4, 8, 16, ⋯.
2. $-\frac{1}{3}$ is the common ratio in the geometric series 27, -9, 3, -1, $\frac{1}{3}$, ⋯.

ORAL EXERCISES

1. Name all the geometric series in the list, Sec. 227.
2. State the common ratio in each of these series.
3. State the 6th term of each of these series.

WRITTEN EXERCISES

1. The series 1, 3, 9, 27, 81, ⋯, whose ratio is 3, is the same as 1, 3^1, 3^2, 3^3, 3^4, ⋯. Write by use of exponents the 6th term of this series; the 8th term; the 10th; the 15th; the 100th.

2. The series 3, $-\frac{3}{2}$, $\frac{3}{4}$, $-\frac{3}{8}$, ⋯, whose ratio is $-\frac{1}{2}$, is the same as 3, $3(-\frac{1}{2})$, $3(-\frac{1}{2})^2$, $3(-\frac{1}{2})^3$, ⋯. Write by use of exponents the 5th term of this series; the 8th term; the 10th; the 25th; the 50th.

239. A general form for the geometric series is:

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}\ \cdots,$$

where a denotes the first term,
r denotes the common ratio, and
n denotes the number of the terms.

240. Last Term. If the last term is numbered n, and denoted by l, then we have for the last of n terms the formula,

$$l = ar^{n-1}.$$

241. The Sum of a Geometric Series. The sum of n terms of a geometric series can readily be found.

EXAMPLE

Find the sum of 5 terms of the series 2, 6, 18, 54, 162.

SOLUTION.

Let	$s = 2 + 6 + 18 + 54 + 162.$	(*1*)
Multiplying by 3, the common ratio,	$3s = 6 + 18 + 54 + 162 + 486.$	(*2*)
Subtracting (*1*) from (*2*),	$3s - s = 486 - 2,$	(*3*)
or,	$2s = 484.$	(*4*)
Dividing by 2,	$s = 242.$	(*5*)

WRITTEN EXERCISES

Find similarly the sum of 5 terms of each of these series:

1. 6, 30, 150, $\cdots$.
2. 7, -14, 28, $\cdots$.
3. $\frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \cdots$.
4. $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, \cdots$.

242. General Formula for the Sum. The general form of the series may be treated in the same way. If l denote the last of n terms, the term before it is denoted by $\frac{l}{r}$, the next preceding by $\frac{l}{r^2}$, and so on. Hence, the sum of n terms may be written:

$$1.\quad s = a + ar + ar^2 + \cdots \frac{l}{r^2} + \frac{l}{r} + l.$$

2. Then $rs = ar + ar^2 + \cdots \frac{l}{r^2} + \frac{l}{r} + l + lr.$

3. Subtracting, $s - rs = a - lr.$

4. Or, $(1 - r)s = a - lr.$

$$\therefore s = \frac{a - lr}{1 - r} = \frac{lr - a}{r - 1}.$$

In words,

The sum of any number of terms of a geometric series is the ratio times the last term diminished by the first term and divided by the ratio less 1.

By using the value of l (Sec. 240),

$$s = \frac{ar^{n-1} \cdot r - a}{r - 1} = \frac{ar^n - a}{r - 1}.$$

Thus, s may be found without first computing l.

WRITTEN EXERCISES

1. Find the sum of 6 terms of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.

2. Find the sum of 10 terms of the series $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \cdots$.

3. Find the sum of 8 terms of the series 1, .25, .0625, ⋯.

4. Find the sum of 12 terms of the series $27, -9, 3, -1, \cdots$.

5. An air pump exhausted the air from a cylinder containing 1 cu. ft. at the rate of $\frac{1}{10}$ of the remaining contents per stroke. What part of a cubic foot of air remained in the cylinder after 25 strokes?

6. The population of a town increased from 10,000 to 14,641 in 5 yr. If the population by years was in geometric series, what was the rate of increase per year?

243. Collected Results. The three chief formulas of geometric series are:

1. $l = ar^{n-1}.$

2. $s = \frac{lr - a}{r - 1}.$

3. $s = \frac{ar^n - a}{r - 1}.$

WRITTEN EXERCISES

1. $100 is placed on interest at 5 %, compounded annually.

(1) What is the amount at the end of the first year?

(2) What is the principal for the second year?

(3) What is the amount at the end of the second year?

1st	\$100 (1.05)
2d	\$100 $(1.05)^2$
3d	\$100 $(1.05)^3$
4th	\$100 $(1.05)^4$

Notice that the amounts appear in the right-hand column of the table.

(4) Indicate similarly the amount of $100 at the end of 5 yr.; 10 yr.; n yr. Which formula of geometric series expresses the amount for n yr.?

2. Indicate the amount of $100 at 6 %, compounded annually, at the end of 1 yr.; 2 yr.; 5 yr.; 10 yr.; n yr.

3. Many savings banks pay interest at the rate of 3 %, compounded semi-annually.

Indicate the amount of $100 under the above conditions at the end of 6 mo.; 1 yr.; 18 mo.; 2 yr.; 10 yr.; n yr.

NOTE. The numerical value of these expressions can be computed readily by logarithms.

Solve by the use of logarithms:

4. What is the amount of $1 at 4 % compound interest for 8 yr.?

5. A man deposits $100 in a bank paying 4 % interest, compounded annually, on the first day of each year for 5 yr. How much will he have on deposit at the end of 5 yr.?

6. Determine similarly the amounts of:

	DEPOSIT AT BEGINNING OF EACH YEAR	RATE OF INTEREST COMPOUNDED ANNUALLY	NUMBER OF YEARS
(1)	$25	5	8
(2)	10	6	15
(3)	43.20	4	20
(4)	39.87	3	20

MEANS

244. Means. Terms standing between two given terms of a series are called **means.**

245. Arithmetical Mean. If three numbers are in arithmetical progression, the middle one is called the **arithmetical mean** between the other two.

The arithmetical mean between a and b is found thus:

1. Let A be the mean and d the common difference.
   Then the terms may be written $A - d$ and $A + d$.
2. Whence, $A - d = a$ and $A + d = b$.
3. Adding, $2A = a + b$ and $A = \frac{a+b}{2}$.

246. *The arithmetical mean between two numbers is one half their sum.*

247. Geometric Mean. If three numbers are in geometric progression, the middle one is called the **geometric mean** between the other two.

The geometric mean between a and b is found thus:

1. Let g be the geometric mean.
2. Then $\frac{g}{a} = \frac{b}{g}$.
3. $\therefore g^2 = ab$ and $g = \sqrt{ab}$.

248. *The geometric mean between two numbers is the square root of their product. There are really two geometric means, one negative and one positive.*

The geometric mean between two numbers is the same as their mean proportional.

ORAL EXERCISES

State the arithmetical mean between:

1. 8, 12. 2. 6, 3. 3. 4, -10. 4. $5a$, $13a$.

State the geometric mean, including signs, between:

5. 8, 6. 6. 3, 12. 7. a, a^5. 8. $2x^3$, $32x^7$.

249. Any number of means may be found by use of formulas already given.

EXAMPLES

1. Insert 5 arithmetical means between 4 and 12.

    1. In this case $a = 4$, $l = 12$, and $n = 7$.
    2. $\therefore l = a + (n - 1)d$ becomes $12 = 4 + (7 - 1)d$.
    3. Solving for d, $d = \frac{12 - 4}{6} = 1\frac{1}{3}$.
    4. Adding $1\frac{1}{3}$ to 4, and $1\frac{1}{3}$ to that result, and so on, the means are found to be $5\frac{1}{3}$, $6\frac{2}{3}$, 8, $9\frac{1}{3}$, and $10\frac{2}{3}$.

2. Insert 4 geometric means between -27 and $\frac{1}{9}$.

    1. In this case $a = -27$, $l = \frac{1}{9}$, and $n = 6$.
    2. $\therefore l = ar^{n-1}$ becomes $\frac{1}{9} = -27r^5$.
    3. Solving for r, $r^5 = -\frac{1}{9 \cdot 27} = -\frac{1}{3^5}$.

    Therefore, $r = -\frac{1}{3}$.

    4. Multiplying -27 by $-\frac{1}{3}$, and multiplying this result by $-\frac{1}{3}$, and so on, the means are found to be 9, -3, 1, and $-\frac{1}{3}$.

WRITTEN EXERCISES

1. Insert 3 arithmetical means between 6 and 26.
2. Insert 10 arithmetical means between -7 and 144.
3. Insert 3 geometric means between 2 and 32.
4. Insert 4 geometric means between $-\frac{1}{10}$ and $3\frac{1}{5}$.

OTHER FORMULAS

250. Arithmetical Series. By means of the formulas of Sec. 236, any two of the five numbers a, n, l, d, s, can be found when the other three are given.

EXAMPLES

1. Given $n = 6$, $s = 18$, $l = 8$, find a, d.

    1. For these values, formulas (1) and (2) become:

$$8 = a + (6 - 1)d,$$

$$18 = \frac{6(a + 8)}{2}.$$

2. We have thus two equations to determine and two numbers, a, d.

   From the second equation, $a = -2$.

3. Using this value in the first equation, $d = 2$.

2. Given $a = 4$, $l = 12$, $s = 56$, find n and d.

1. For these values formulas (1) and (2) become:

$$12 = 4 + (n-1)d.$$

$$56 = \frac{n(4+12)}{2}.$$

2. $\therefore$ from the second equation, $n = 7$.
3. Substituting in the first, $12 = 4 + 6\,d$,
4. therefore $d = \frac{4}{3}$.

3. Given $n = 12$, $s = 30$, $l = 10$, find a, d.

1. Formulas (1) and (3) become:

$$10 = a + 11\,d.$$

$$30 = \frac{12(2\,a + 11\,d)}{2} = 12\,a + 66\,d.$$

2. Solving these equations for a and d:

$$a = -5, \text{ and } d = \tfrac{15}{11}.$$

The same results would be found by using formulas (1) and (2), since (3) is only another form of (2).

251. The same problems can also be solved generally; that is, without specifying numerical values.

EXAMPLE

Regarding n, s, d as known, find a, l.

1. From (1), Sec. 236, $l - a = (n-1)d$.
2. From (2), Sec. 236, $a + l = \frac{2\,s}{n}$.
3. Adding (1) and (2), $2\,l = (n-1)d + \frac{2\,s}{n}$.

   Or, $l = \frac{(n-1)d}{2} + \frac{s}{n}$.

4. Substituting in (2) above, $a = \frac{2\,s}{n} - \left[\frac{(n-1)d}{2} + \frac{s}{n}\right]$

$$= \frac{s}{n} - \frac{(n-1)d}{2}.$$

WRITTEN EXERCISES

By use of the formulas in Sec. 236, find the following:

	Find	In Terms of	Result
1.	l	$a\,d\,n$	$l = a + (n-1)\,d$
2.	l	$a\,d\,s$	$l = \frac{1}{2}[-d \pm \sqrt{8\,ds + (2\,a - d)^2}]$
3.	l	$a\,n\,s$	$l = \frac{2\,s}{n} - a$
4.	l	$d\,n\,s$	$l = \frac{s}{n} + \frac{(n-1)\,d}{2}$
5.	s	$a\,d\,n$	$s = \frac{1}{2}\,n\,[2\,a + (n-1)\,d]$
6.	s	$a\,d\,l$	$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2\,d}$
7.	s	$a\,n\,l$	$s = \frac{n}{2}(a + l)$
8.	s	$d\,n\,l$	$s = \frac{1}{2}\,n\,[2\,l - (n-1)\,d]$
9.	a	$d\,n\,l$	$a = l - (n-1)\,d$
10.	a	$d\,n\,s$	$a = \frac{s}{n} - \frac{(n-1)\,d}{2}$
11.	a	$d\,l\,s$	$a = \frac{1}{2}[d \pm \sqrt{(2\,l + d)^2 - 8\,ds}]$
12.	a	$n\,l\,s$	$a = \frac{2\,s}{n} - l$
13.	d	$a\,n\,l$	$d = \frac{l-a}{n-1}$
14.	d	$a\,n\,s$	$d = \frac{2\,(s - an)}{n\,(n-1)}$
15.	d	$a\,l\,s$	$d = \frac{l^2 - a^2}{2\,s - l - a}$
16.	d	$n\,l\,s$	$d = \frac{2\,(nl - s)}{n\,(n-1)}$
17.	n	$a\,d\,l$	$n = \frac{l-a}{d} + 1$
18.	n	$a\,d\,s$	$n = \frac{d - 2\,a \pm \sqrt{(2\,a - d)^2 + 8\,ds}}{2\,d}$
19.		$a\,l\,s$	$n = \frac{2\,s}{l+a}$
20.	n	$d\,l\,s$	$n = \frac{2\,l + d \pm \sqrt{(2\,l + d)^2 - 8\,ds}}{2\,l}$

Note. a, l, d, s may have any values, but n must be a positive integer. Hence, not all solutions of equations 18, 20, represent possible series.

252. Geometric Series. By means of the formulas of Sec. 243, any two of the five numbers a, n, l, r, s, can be found when the other three are given.

EXAMPLES

1. Given $s = 1024$, $r = 2$, $a = 2$, find l.

   1. For these values formula (2) becomes:

   $$1024 = \frac{2\,l - 2}{2 - 1} = 2(l - 1).$$

   2. $\therefore l = 513$.

2. Given $r = 5$, $n = 5$, $s = 363$, find a.

   1. For these values formulas (1) and (2) become:

   $$l = a \cdot 3^4.$$
   $$363 = \frac{3\,l - a}{2}.$$

   2. Eliminating l, $363 = \dfrac{a(3^5 - 1)}{2}$.

   3. Therefore, $363 = \frac{1}{2}\,a \cdot 242$, and $a = 3$.

3. Given $s = 363$, $a = 3$, $r = 3$, find n.

   1. For these values formula (3) becomes:

   $$363 = \frac{3 \cdot 3^n - 3}{2}.$$

   2. Therefore, $3^n - 1 = 242$, and $3^n = 243$.

   3. By factoring 243, n is seen to be 5.

NOTE. In finding n it may not be possible to factor as in the case of 243 above. In this case logarithms may be applied.

253. The same problems can be solved generally, that is, without specifying numerical values.

EXAMPLE

Express l in terms of a, n, and s.

1. From formula (2) $r = \dfrac{s - a}{s - l}$, or $r^{n-1} = \dfrac{(s - a)^{n-1}}{(s - l)^{n-1}}$.

2. $\therefore$ substituting in (1) $l = \dfrac{a(s - a)^{n-1}}{(s - l)^{n-1}}$.

3. $\therefore l(s - l)^{n-1} - a(s - a)^{n-1} = 0$.

This equation is of a degree higher than 2 in l when $n > 3$. But for n equal to or less than 3 it can be solved by methods already explained.

WRITTEN EXERCISES

By use of the formulas in Sec. 243, find the results given in the table:

Note. In Exercises 3, 12, and 16, only the equation connecting the unknown numbers with the given ones can be found:

	Find	In Terms of	Result
1.	l	$a\ r\ n$	$l = ar^{n-1}$
2.	l	$a\ r\ s$	$l = \frac{a + (r-1)s}{r}$
3.	l	$a\ n\ s$	$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$
4.	l	$r\ n\ s$	$l = \frac{(r-1)sr^{n-1}}{r^n - 1}$
5.	s	$a\ r\ n$	$s = \frac{a(r^n - 1)}{r - 1}$
6.	s	$a\ r\ l$	$s = \frac{rl - a}{r - 1}$
7.	s	$a\ l\ n$	$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$
8.	s	$r\ n\ l$	$s = \frac{lr^n - l}{r^n - r^{n-1}}$
9.	a	$r\ n\ l$	$a = \frac{l}{r^{n-1}}$
10.	a	$r\ n\ s$	$a = \frac{(r-1)s}{r^n - 1}$
11.	a	$r\ l\ s$	$a = rl - (r-1)s$
12.	a	$n\ l\ s$	$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$
13.	r	$a\ n\ l$	$r = \sqrt[n-1]{\frac{l}{a}}$
14.	r	$a\ n\ s$	$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0$
15.	r	$a\ l\ s$	$r = \frac{s-a}{s-l}$
16.	r	$n\ l\ s$	$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0$

SPECIAL SERIES

254. Binomial Expansion. The Binomial Expansion, Chapter XX, Part I, is a series in which each term of $(a+b)^n$ is produced from the next preceding one by multiplying by $\frac{b}{a}$ and inserting in the numerator and the denominator the next factor in each sequence.

For example, the third term in the binomial expansion of $(a+b)^n$ is $\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$, and the fourth term is

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3.$$

To form the fourth term from the third, note that:

1. $a^{n-2}b^2 \times \frac{b}{a} = a^{n-3}b^3$.

2. The numerator, $n(n-1)(n-2)$, of the coefficient of the fourth term has one more factor in the sequence of factors, which begins with n and decreases by 1 each time.

3. The denominator, $1\cdot 2\cdot 3$, has one more factor in the sequence of factors which begins with 1 and increases by 1 each time.

4. In the expansion of $(a-b)^n$, the even terms are negative.

Thus, from any given term of the binomial series, all the subsequent terms can be written.

WRITTEN EXERCISES

1. The fifth term of a binomial expansion is

$$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}y^4.$$

Write the sixth term; also the seventh term.

2. The sixth term of a binomial expansion is

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{n-5}b^5.$$

Find the eighth term.

3. By Sec. 325, Part I, $1\cdot 2$ may be written $\underline{|2}$, called *factorial two,* $1\cdot 2\cdot 3$ may be written $\underline{|3}$, called *factorial three,* and so on.

Write the denominators of the coefficients in Exercise 1 as factorials. Also in Exercise 2.

4. Suppose n is 8 in Exercise 1. What do the sixth and seventh terms become? Suppose n is 10 in Exercise 2, what does the eighth term become?

255. Finding the *r*th Term. Any term of the binomial series may be written at once by observing the general form of the terms, as explained in Sec. 254.

Observe that:

1. *The last factor in the numerator of each coefficient is* n *minus a number two less than the number of the term.*

E.g., in the third term the last factor is $n-1$; in the 4th it is $n-2$; in the 5th, $n-3$, and so on.

2. *The denominator is the factorial of the number one less than the number of the term.*

E.g., $\underline{|2}$ in the third, $\underline{|3}$ in the 4th, and so on

3. *The exponent of the first term of the binomial, or* a *in* $(a \pm b)^n$, *is* n *minus a number one less than the number of the term;*

E.g., $n-1$ in the 2d term, $n-2$ in the 3d, and so on.

4. *The exponent of* b *is one less than the number of the term;*

E.g., 1 in the 2d term, 2 in the 3d, and so on.

5. *The signs of all terms are positive in* $(a+b)^n$. *In* $(a-b)^n$ *the odd terms are positive, and the even terms negative.*

EXAMPLES

1. Find the 9th term of $(x-y)^p$.

By (1) the numerator of the coefficient is $p(p-1)(p-2)\cdots(p-7)$.
By (2) the denominator of the coefficient is $\underline{|8}$.
By (3) the exponent of x is $p-8$.
By (4) the exponent of y is 8.
By (5) the sign is $+$.
Therefore, the 9th term of $(x-y)^p$ is

$$+\frac{p(p-1)(p-2)\cdots(p-7)}{\underline{|8}}x^{p-8}y^8.$$

2. Find the rth term of $(x - 2a)^n$.

By (1) the numerator of the coefficient is $n(n-1)\cdots(n-\overline{r-2})$.
By (2) the denominator of the coefficient is $\underline{|r-1}$.
By (3) the exponent of x is $n-\overline{r-1}$.
By (4) the exponent of $(2a)$ is $r-1$.
By (5) the sign is $\pm$, according as r is odd or even.
Therefore, the rth term of $(x-2a)^n$ is

$$\pm\frac{n(n-1)\cdots(n-r+2)}{\underline{|r-1}}x^{n-r+1}(2a)^{r-1}$$

WRITTEN EXERCISES

Expand:

1. $(x-a)^5$. 2. $(a+x)^6$. 3. $(2a-x)^7$.

Write without expanding the series:

4. The 5th term of $(a-2b)^7$.
5. The 7th term of $(a-x)^{10}$.
6. The middle term of $(a+x)^{12}$.
7. The two middle terms of $(2x-y^2)^{13}$.
8. The 10th term of $(a+3x^3)^{21}$.
9. The qth term in $(\frac{1}{2}x-2y)^n$.
10. The rth term in $(x-y)^{\frac{1}{2}}$.
11. The $(r+1)$st term in $(a+b)^n$.
12. The $(r-5)$th term in $(a-b)^{n+1}$.

256. Finite Series. So far we have treated only series with a fixed number of terms. A series which comes to an end is called a **finite series.**

257. Infinite Series. A series whose law is such that every term has a term following it is called an **infinite series.**

For example:

2, 5, 8, 11, ⋯, 239 as here written ends with 239. But the law of the series would permit additional terms to be specified. In the above example, the next following terms would be 242, 245, etc. It is obvious that however many terms may have been specified, still more can be made by adding 3. The series is thus unending.

Similarly, all of the series so far considered might have been continued by applying their corresponding laws.

The term "infinite" comes from the Latin *infinitus*, and is here used with the meaning *unending*.

If the coefficients of the binomial expansion be regarded as a series,

$$1,\ n,\ \frac{(n-1)}{2\,!},\ \frac{n(n-1)(n-2)}{3\,!},\ \frac{n(n-1)(n-2)(n-3)}{4\,!},\ \cdots$$

they furnish, when n is a positive integer, instances of series that come to an end according to the law of the series. If $n = 3$, the series has 4 terms, and if $n = 10$, the series has 11 terms; for the positive integer n, it has $n + 1$ terms. This is true because the factors n, $n - 1$, $n - 2$, and so on, will finally in the $(n + 2)$d term contain $n - n$ or zero. Therefore the series has $n + 1$ terms.

But if n is a negative integer or a fraction, none of the factors, n, $n - 1$, $n - 2$, and so on, becomes zero, and the series can always be extended farther. That is, if n is a negative integer or any fraction, the series is unending or infinite.

258. Infinite Geometric Series. The subject of infinite series is of great importance, but is too difficult to be taken up here. We shall mention simply a few properties of infinite geometric series *whose ratio is numerically less than* 1.

The following are examples of such series:

1. $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \cdots.$
2. $3,\ \frac{3}{5},\ \frac{3}{25},\ \frac{3}{125},\ \cdots.$
3. $.5,\ .05,\ .005,\ .0005,\ \cdots.$
4. $1,\ -\frac{1}{2},\ \frac{1}{4},\ -\frac{1}{8},\ \frac{1}{16},\ -\frac{1}{32},\ \cdots.$

State the ratio and the next three terms of each series.

I. *The terms become numerically smaller and smaller.* Each term is numerically smaller than the one preceding it, for it is a fractional part of it.

II. *The terms become numerically small at will.* That is, however small a number may be selected, there are terms in the series smaller than it, and when r is numerically less than 1, the term ar^{n-1} may be made numerically small at will, by taking n sufficiently large.

This seems obvious from the consideration of the series given above as examples. It is not difficult to prolong these series until their terms are less than $\frac{1}{100}$ say, or $\frac{1}{1000}$, and from this it seems plausible to think that the terms would become less than one millionth, or one billionth, or any other number, if a sufficient number of terms are taken. As a matter of fact this is true, but the proof is too difficult to be given here.

III. We have proved that, if s_n denote the sum of the first n terms of a geometric series,

$$s_n = \frac{a - ar^n}{1 - r}.$$

This may be written: $s_n = \frac{a}{1-r} - ar^{n-1}\left(\frac{r}{1-r}\right)$.

By taking n sufficiently large, the product of ar^{n-1} and the fixed number $\frac{r}{1-r}$ can be made as small as desired. As more and more terms of the series are added, the sum differs less and less from $\frac{a}{1-r}$; and if sufficient terms are taken, the sum comes and remains as close as we please to $\frac{a}{1-r}$.

The number $\frac{a}{1-r}$ is called the **limit** of the sum of n terms, as n is increased without bound. Denoting this limit by s, we have:

$$s = \frac{a}{1-r}.$$

The number s is not the sum of *all* the terms of the series, for since the terms of the series never come to an end, the operation of adding them cannot be completed.

According to Sec. 258,

$$s \doteq \frac{a}{1-r} \text{ when } n = \infty.$$

For example:

When $a = 4$, and $r = \frac{1}{2}$, then the limit of $s = \frac{4}{1 - \frac{1}{2}} = 8$.

To test this, we form successive values of s_n.

$$s_1 = 4.$$
$$s_2 = 6.$$
$$s_3 = 7.$$
$$s_4 = 7\tfrac{1}{2}.$$
$$s_5 = 7\tfrac{3}{4}.$$
$$s_6 = 7\tfrac{7}{8}.$$

It appears that the values of s_n approximate more and more closely to 8 as n is increased.

WRITTEN EXERCISES

Find the limit of the sum of the series:

1. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

3. $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} \cdots$.

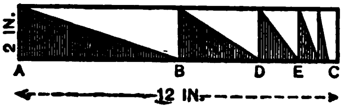
4. $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \cdots$.

5. Test the results of the preceding exercises by finding successive values of s_n.

6. In an infinite geometric series $s = 2$ and $r = \frac{1}{2}$; find a.

7. Find the fraction which is the limit of $.\dot{3}33333 \cdots$, or $.3 + .03 + .003 + \cdots$.

8. Find the limit of $.\dot{2}\dot{3}232323 \cdots$ or $.23 + .0023 + .000023 + \cdots$.



9. Triangles are drawn in a rectangle of dimensions indicated, B being the midpoint of AC, D that of BC, E that of DC, and so on. What limit does the sum of the areas of the triangles approach as more and more triangles are taken?

10. Find the sum of 16 terms of the series,

$$27,\ 22\tfrac{1}{2},\ 18,\ 13\tfrac{1}{2},\ \cdots.$$

11. Find the sum of 18 terms of the series,

$$36,\ 12,\ 4,\ \tfrac{4}{3},\ \cdots.$$

12. The difference between two numbers is 48. Their arithmetical mean exceeds their geometric mean by 18. Find the numbers.

13. Express as a geometric series the decimal fraction.

$$.0373737 \cdots.$$

What is its limiting value?

14. Find the limiting value of each of these series:

(a) $.\dot{3}\dot{5}3535 \cdots$.

(b) $.125\dot{6}66 \cdots$.

(c) $.03\dot{2}\dot{4}24 \cdots$.

(d) $.\dot{1}2\dot{5}125 \cdots$.

(e) $3.\dot{6}\dot{0}\dot{5}605 \cdots$.

(f) $5.00\dot{8}88 \cdots$.

(g) $9.\dot{1}\dot{0}1010 \cdots$.

(h) $6.04\dot{3}\dot{8}38 \cdots$.

15. If $\frac{1}{b-a}, \frac{1}{2b}, \frac{1}{b-c}$, are in arithmetical progression, show that a, b, c are in geometric progression.

SUGGESTION. The supposition means that

$$\frac{1}{b-a} - \frac{1}{2b} = \frac{1}{2b} - \frac{1}{b-c}.$$

This reduces to $b^2 = ac$.

16. Find the amount in n years of P dollars at r per cent per annum, interest being compounded annually.

17. During a truce, a certain army A loses by sickness 14 men the first day, 15 the second, 16 the third, and so on; while the opposing army B loses 12 men every day. At the end of fifty days the armies are found to be of equal size. Find the difference between the two armies at the beginning of the truce.

18. A strip of carpet one half inch thick and $29\frac{6}{7}$ feet long is rolled on a roller four inches in diameter. Find how many turns there will be, remembering that each turn increases the diameter by one inch, and taking as the length of a circumference $\frac{22}{7}$ times the diameter.

19. Insert between 1 and 21 the arithmetic means such that the sum of the last three terms of the series is 48.

20. If $\frac{a}{b} = \frac{c}{d}$, prove that $ab + cd$ is a mean proportional between $a^2 + c^2$ and $b^2 + d^2$.

21. The sum of the first ten terms of a geometric series is 244 times the sum of the first five terms; and the sum of the fourth and sixth terms is 135. Find the first term and the common ratio.

REVIEW

WRITTEN EXERCISES

1. Find the 47th multiple of 7.
2. Find the sum of the first 12 multiples of 4.

Find the 20th term, and the sum of 12 terms of each series:

3. $6, 9, \frac{27}{2}, \frac{81}{4}, \cdots$.

4. $8, 11, 14, 17, \cdots$.

5. $2^9, 2^6, 2^3, \cdots$.

6. $a+b, a-b, a-3b, a-5b$.

Find the eighth term, and the sum of 8 terms:

7. $1, 4, 16, \cdots$.

8. $3, 6, 12, \cdots$.

9. $2, -4, 8, -16, \cdots$.

10. $1, -2, 2^2, -2^3, \cdots$.

11. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \cdots$.

12. $100, -40, 16, \cdots$.

Find the twelfth term, and the sum of 12 terms:

13. $2, 4, 6, \cdots$.

14. $-5, -3, -1, \cdots$.

15. $1, \frac{6}{7}, \frac{5}{7}, \cdots$.

16. $\frac{2}{3}, \frac{7}{15}, \frac{4}{15}, \cdots$.

17. $4, -3, -10, \cdots$.

18. $\frac{1}{2}, -\frac{2}{3}, -\frac{11}{6}, \cdots$.

19. Find three numbers whose common difference is 1 and such that the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

20. The first term of an arithmetical series is $n^2 - n - 1$, the common difference is 2. Find the sum of n terms.

21. In some countries of Europe the hours of the day are numbered on the clockface from 1 to 24. How many strokes would a clock make per day in striking these hours?

22. How many strokes does a common clock striking the hours make in a day?

23. A man leases a business block for 20 years under the condition that, owing to estimated increase in the value of the property, the rental is to be increased $50 each year. He pays altogether $19,500. What was the rental of the first year? The last?

24. A railroad car starting from rest began to run down an inclined plane. It is known that in such motion the distances passed over in successive seconds are in arithmetical progression. It was observed that at the end of 10 sec. the car had passed over 570 ft. and at the end of 20 sec. 2340 ft. from the starting point. How far did it run the first second? How far from the starting point was it at the end of 15 sec.?

25. It is known that if a body falls freely, the spaces passed over in successive seconds are in arithmetical progression, and that it falls approximately 16 ft. in the first second and 48 ft. in the next second. To determine the height of a tower, a ball was dropped from the top and observed to strike the ground in 4 sec. Find the height of the tower.

26. An employee receives a certain annual salary, and in each succeeding year he receives \$72 more than the year before. At the end of the tenth year he had received all together \$10,440. What was his salary the first year? The last?

27. The 14th term of an arithmetical series is 72, the fifth term is 27. Find the common difference and the first term.

28. A man is credited \$100 annually on the books of a building society as follows: At the beginning of the first year he pays in \$100 cash. At the beginning of the second year he is credited with \$6 interest on the amount already to his credit; and he is required to pay \$94 in cash, making his total credit \$200. At the beginning of the third year he is credited with \$12 interest, and pays \$88 in cash, and so on. How much is his payment at the beginning of the tenth year? What is his credit then? How much cash has he paid in all?

29. At each stroke an air pump exhausts $\frac{2}{3}$ of the air in the receiver. What part of the original air remains in the receiver after the 8th stroke?

30. At the close of each business year, a certain manufacturer deducts 10 % from the amount at which his machinery was valued at the beginning of the year. If his machinery cost \$10,000, at what did he value it at the end of the fourth year?

31. In Exercise 30, by use of logarithms, find its valuation at the end of the 20th year.

32. Show that the terms of a geometric progression form a continued proportion, by applying Sec. 209 to the series a, ar, ar^2, ar^3, ar^4, $\cdots$.

33. We have shown in Sec. 258 that $\frac{a}{1-r}$ is the limit of the sum of the terms of a geometric progression whose first term is a and whose common ratio is r. Find the terms of this series by dividing the numerator of the above fraction by its denominator.

34. What kind of a series do the reciprocals of the numbers 2, 3, and 6 form?

35. Find the arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$.

36. Find the sum of each of the following infinite series:

(1) $3 + \frac{1}{2} + \frac{1}{12} + \cdots$.
(2) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$.
(3) $1 + \frac{4}{5} + \frac{16}{25} + \cdots$.

37. Find the sum of 10 terms of each series in Exercise 36.

38. Write the *rth* term of the binomial expansion; also the $(r+1)$*st* term. Find the ratio of the *rth* term to the $(r+1)$*st* term, and simplify the result.

39. Write the term which contains x^{10} in the expansion of $(a-x)^{12}$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. When is a group of numbers called a *series?* Sec. 226.
2. What is meant by the *terms* of a series? Sec. 227.
3. Define and illustrate an *arithmetical series*, or *progression.* Sec. 230.
4. Define *common difference.* Sec. 231.
5. Define and illustrate a *geometric progression.* Sec. 237.
6. Define *common ratio.* Sec. 238.
7. Define *arithmetical mean; geometric mean.* Secs. 245–248.

8. State the formula for the *last term* or general term of an arithmetical progression; also of a geometric progression. Secs. 233, 240.

9. State the formula for the sum of n terms of an arithmetical progression; also of a geometric progression. Secs. 234, 241.

10. State a formula to find n in terms of a, l, and s in an arithmetical progression. Sec. 251.

11. State a formula to find d in terms of a, n, and l in an arithmetical progression. Sec. 251.

12. State a formula to find a in terms of n, l, and s in an arithmetic progression. Sec. 251.

13. State a formula to determine a in terms of r, n, and l in a geometric progression. Sec. 253.

14. State a formula to determine r in terms of a, l, and s in a geometrical progression. Sec. 253.

15. State the general form of the rth term of the binomial expansion $(a \pm b)^n$. Sec. 255.

16. Define and illustrate an *infinite series*. Sec. 257.

17. State the expression for the limit of s in an infinite geometric series. Sec. 258, III.

HISTORICAL NOTE

The existence of sets of successive numbers, each term of which depends, in a definite way, upon its predecessors for its value, called a progression, or series of numbers, was discovered by the early mathematicians. One of the two Babylonian tablets still in existence gives in cuneiform symbols the squares of the integers from 1 to 60, namely, 1, 4, 9, 16, 25, and so on. The other tablet gives the following numbers: 5, 10, 20, 40, 80, 96, 112, and so on, the first five of which form a geometric progression and the rest an arithmetical series. These numbers were used to represent the illuminated portions of the moon's disk from day to day, from new moon to full moon. Thus, taking the whole as 240 parts, the visible portion of the moon for the first day would be $\frac{5}{240}$, or $\frac{1}{48}$ of the whole disk.

The terms of a geometric series with an integral ratio increase at a very rapid rate, and the earlier mathematicians seem to have taken much interest in the framing of problems intended to exemplify this. Sixteenth-century writers of arithmetic collected many of these problems, of which the following are typical:

A peasant agreed with a blacksmith to pay him for shoeing his horse at the rate of one cent for the first nail, two cents for the second, four cents for the third, and so on, in geometric progression. There being 8 nails in each of the four shoes, how much was the peasant to pay the blacksmith?

Required the number of kernels of wheat needed in order to place one kernel on the first square of the chessboard, two on the second, four on the third, and so on, for the sixty-four squares. This later problem was given by Masudi, in *Meadows of Gold* (950 A.D.).

The process of summing arithmetical and geometric series (Secs. 235, 242), and the methods for finding any required term, were known to the Hindoos and appear in the works of Bhaskara. But these were trifling beginnings compared with the part played by the almost unlimited variety of series used in modern mathematics. The proof of the binomial expansion by Newton, the ability to express logarithms by series, accomplished by Mercator (1668), and the study of other systems of infinite series has opened vast new fields of mathematics.

CHAPTER X

GEOMETRIC PROBLEMS FOR ALGEBRAIC SOLUTION

The problems in the following list may be used as supplementary work for pupils that have studied plane geometry. In the body of the Algebra numerous problems have been given applying geometric facts which the pupil has learned in the study of mensuration in arithmetic. In the following list the problems contain the application of other relations and theorems of geometry, and typical solutions have been inserted to suggest to the pupil the method of attack.

LINEAR EQUATIONS. ONE UNKNOWN

1. In a given triangle one angle is twice another, and the third angle is 24°. Find the unknown angles.

SOLUTION. Let x be one of the unknown angles,

then $2x$ is the other. *(1)*

Because the sum of the angles of a triangle = 180°,

$$x + 2x + 24° = 180°. \qquad (2)$$

Solving equation (2)

$$x = 52° \text{ and } 2x = 104°. \qquad (3)$$

The angles of the triangle are 52°, 104°, 24°.

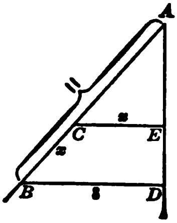
2. In a certain triangle one angle is three times another, and the third angle is 36°. Find the unknown angles.

3. In a given right-angled triangle one acute angle is $\frac{2}{3}$ the other. Find the angles.

4. In a certain isosceles triangle the angle opposite to the base is 18°. Find the angles at the base.

5. The three angles A, B, and C of a given triangle are in the ratio of 2, 3, and 5. Find the angles.

6. Given an angle A such that a point B situated on one side 11 in. from the vertex is 8 in. distant from the other side. Find a point C on the same side as B, and equidistant from B and the other side of the angle.



Solution. Let A be the given angle, then the figure represents the conditions of the problem.

From the similar triangles ACE and ABD, we have

$$\frac{AC}{CE} = \frac{AB}{BD},$$

or, $$\frac{11 - x}{x} = \frac{11}{8}. \qquad (1)$$

From (1), $$88 - 8x = 11x. \qquad (2)$$

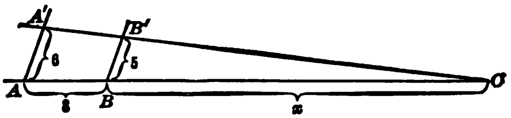
Then, $$88 = 19x \text{ and } x = \frac{88}{19}. \qquad (3)$$

The distance CB is $\frac{88}{19}$ in., or $4\frac{12}{19}$.

7. Solve problem 6, if the point B is 9 in. from A and 6 in. from side AD.

8. Solve problem 6, if the point B is a in. from the vertex of the angle and b in. from the other side of the angle.

9. Two points A and B are 8 in. apart. Parallels are drawn through A and B; on these parallels the points A' and B' are located on the same side of the straight line through AB and at distances 6 in. and 5 in. from A and B, respectively. Determine the point where the line $A'B'$ cuts the line AB.



Solution. Let C be the desired point and let $BC = x$. Then, by similar triangles,

$$\frac{BC}{BB'} = \frac{AC}{AA'}, \qquad (1)$$

or, $$\frac{x}{5} = \frac{x + 8}{6}. \qquad (2)$$

Hence, $6x = 5x + 40.$ (*3*)

and $x = 40.$ (*4*)

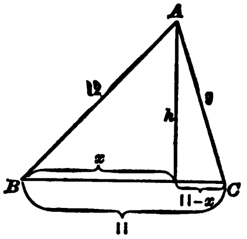
The point is 40 in. from *B*.

10. Solve the same problem if *A'* and *B'* lie on opposite sides of *AB*.

11. Solve the same problem if the distance *AB* is *d*, and the points *A'*, *B'* lie on the same side of *AB*, and at distances *a* and *b* from *A* and *B*, respectively, with $a > b$.

12. Solve the preceding problem if the points *A'* and *B'* lie on opposite sides of *AB*.

13. The three sides of a triangle are 11, 9, 12. A perpendicular is dropped on the side of length 11 from the opposite vertex. Find the lengths of the segments into which the foot of the perpendicular divides that side.



Solution. Using the notations of the figure,

$$h^2 = 12^2 - x^2, \quad (1)$$

and $$h^2 = 9^2 - (11 - x)^2. \quad (2)$$

From (*1*) and (*2*), $$12^2 - x^2 = 9^2 - (11 - x)^2. \quad (3)$$

Rearranging (*3*), $$12^2 - 9^2 + 11^2 = 22x. \quad (4)$$

Solving (*4*), $$x = \tfrac{92}{11}. \quad (5)$$

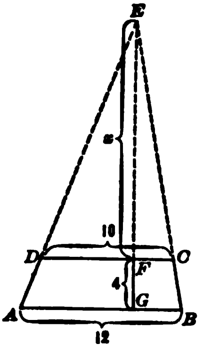
The other segment is $11 - x$, or $\tfrac{29}{11}$. (*6*)

The segments are $\tfrac{92}{11}$ and $\tfrac{29}{11}$.

14. Solve the preceding problem if the sides are 4, 7, 9, and the perpendicular is dropped on the side of length 7.

15. Solve the same problem if the sides of the triangle are *a*, *b*, *c*, and the perpendicular is dropped on the side of length *a*.

16. The lower base of a trapezoid is 12, the upper base is 10, and the altitude is 4. Determine the altitude of the triangle formed by the upper base and the prolongation of the two non-parallel sides until they meet.



Solution. Using the notations of the figure,

$$\frac{EF}{EG}=\frac{DC}{AB}, \tag{1}$$

or

$$\frac{x}{x+4}=\frac{10}{12}. \tag{2}$$

Hence,

$$12x=10x+40, \tag{3}$$

or

$$x=20. \tag{4}$$

The altitude is 20.

17. Solve the same problem if the lower base is a, the upper base b, and the altitude h.

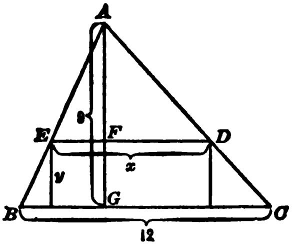
LINEAR EQUATIONS. TWO UNKNOWNS

18. A rectangle 5 in. longer than it is wide is inscribed in a triangle of base 12 in. and altitude 9 in., the longer side resting on the base of the triangle. Find the dimensions of the rectangle.

Solution Let x denote the longer side and y the shorter.

Then,

$$x-y=5. \tag{1}$$



In the similar triangles ABC and AED,

$$\frac{AF}{AG}=\frac{ED}{BC}, \tag{2}$$

or

$$\frac{9-y}{9}=\frac{x}{12}. \tag{3}$$

From (*3*), $108 - 12y = 9x.$ (*4*)

From (*1*) and (*4*), $108 - 12y = 9(y + 5),$ (*5*)

or $21y = 63.$ (*6*)

$y = 3.$ (*7*)

From (*7*) and (*1*), $x = 8.$ (*8*)

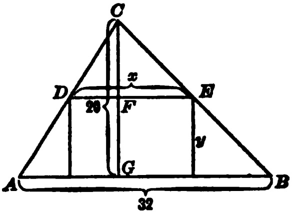
The dimensions of the rectangle are 3 in. and 8 in.

19. Solve the preceding problem if the shorter side rests on the base of length 12 in.

20. Solve Problem 18 if the difference of the sides is *d*, the length of the base of the triangle is *a*, the altitude is *h*, and the longer side of the rectangle rests on the base of the triangle.

21. Solve the preceding problem if the shorter side of the rectangle rests on the given base of the triangle.

22. A rectangle similar to a rectangle whose sides are 5 and 8 is inscribed in a triangle of base 32 and altitude 20. The longer side of the rectangle rests on the given base of the triangle. Find the dimensions of the rectangle.



SOLUTION. Let x and y denote the sides of the inscribed rectangle. Then from the similar triangles EDC and ABC,

$$\frac{DE}{AB} = \frac{CF}{CG}, \qquad (1)$$

or

$$\frac{x}{32} = \frac{20 - y}{20}. \qquad (2)$$

From the similarity of the rectangles,

$$\frac{x}{y} = \frac{8}{5}. \qquad (3)$$

From (*3*), $x = \frac{8}{5}y.$ (*4*)

From (*2*), $20x = 640 - 32y.$ (*5*)

From (*4*) and (*5*), $32y = 640 - 32y,$ (*6*)

or $64y = 640.$ (*7*)

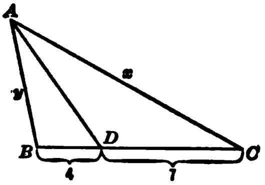
$y = 10.$ (*8*)

From (*8*) and (*4*), $x = 16.$ (*9*)

23. Solve the preceding problem if the shorter side of the rectangle rests on the given side of the triangle.

24. Solve the same problem if the given side of the triangle is of length a, and the altitude on it is of length h, and the given rectangle has dimensions l and m, provided the inscribed rectangle has its side corresponding to the side l of the given rectangle resting on the given base of the triangle.

25. The bisector of an angle of a given triangle divides the side opposite to the angle into two segments of lengths 4 in. and 7 in. The difference between the other two sides of the triangle is 5 in. Find the perimeter of the triangle.



SOLUTION. Let ABC be the given triangle, AD the bisector of angle A, and x and y the required sides.

Then, $x - y = 5$, given in the problem, (1)

and $\frac{x}{y} = \frac{7}{4}$, by geometry the bisector divides the opposite side into segments proportional to the adjacent sides. (2)

$4x = 7y$, from (2). (3)

$7y - 4y = 20$, from (3) and 4 times (1). (4)

Then, $y = 6\frac{2}{3}$, solving (4). (5)

Then, $x = 11\frac{2}{3}$, from (5) and (1). (6)

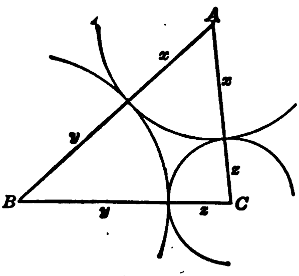
The perimeter is 11 in. + $6\frac{2}{3}$ in. + $11\frac{2}{3}$ in. = $29\frac{1}{3}$ in.

26. Solve the preceding problem, if the segments of the base are l and m and the difference between the sides is d. Also, if the segments of the base are p and q and the difference between the sides is r.

27. The sides of a triangle are 8 ft., 12 ft., and 15 ft., and the angle between the sides 8 and 12 is bisected by a line cutting the side 15. What is the length of each segment of the line 15?

LINEAR EQUATIONS. THREE UNKNOWNS

28. The points A, B, and C are situated so that $AB = 8$ in., $BC = 6$ in., $AC = 5$ in. Find the radii of three circles having the three points as centers and each tangent to the other two externally.



SOLUTION. Let x, y, z, be the radii of the three circles as indicated in the figure.

Then, $x + y = 8$, (1)

$x + z = 5$, (2)

$y + z = 6$. (3)

Adding (1), (2), and (3), and dividing the result by 2,

$x + y + z = \frac{19}{2}$. (4)

From (3) and (4), $x = \frac{7}{2}$. (5)

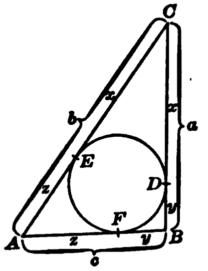
From (2) and (4), $y = \frac{9}{2}$. (6)

From (1) and (4), $z = \frac{3}{2}$. (7)

The radii are $3\frac{1}{2}$ in., $4\frac{1}{2}$ in., and $1\frac{1}{2}$ in.

29. Solve the same problem if $AB = 12$, $BC = 16$, and $AC = 20$.

30. Solve Problem 28, if $BC = a$, $AC = b$, $AB = c$.



31. A triangle ABC is circumscribed about a circle and its sides are tangent at points E, D, F. The sides of the triangle are $a = 8$ in., $b = 15$ in., and $c = 12$ in. Find the segments into which points E, D, F divides the sides.

SOLUTION. The tangents from an external point are equal, hence in the figure,

$x + y = 8$, (1)

$x + z = 15$, (2)

$y + z = 12$. (3)

Subtracting (3) from (2), $x - y = 3$. (4)

Adding (1) and (4) $2x = 11$ (5)

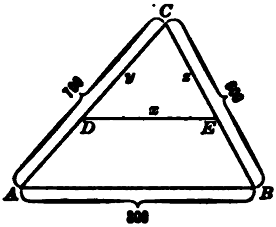
$\therefore\ x = 5\frac{1}{2}$.

From (1) and (2) $y = 2\frac{1}{2}$ and $z = 9\frac{1}{2}$.

32. Solve the same problem, if $a = 24$ ft., $b = 10$ ft., and $c = 24$ ft.

Also, if $a = 30$ yd., $b = 40$ yd., $c = 50$ yd.

33. A man owned a triangular, unfenced field of sides 600 yd., 700 yd., and 800 yd. He sold a triangular piece cut off by a straight line parallel to the side of length 800 yd. and found that 1800 running yards of fence would be required to inclose what remained. Find the lengths of the sides of the portion sold.



Solution. Using the notations of the figure,

$$AB + BC + AC + 2\,DE = AB + BE + EC + CD + DA + 2\,DE$$
$$= (AB + BE + ED + DA) + (EC + CD + DE). \quad (1)$$

Hence, $$2100 + 2x = 1800 + (x + y + z). \quad (2)$$

But $$\frac{CD}{CA} = \frac{DE}{AB}, \quad (3)$$

and $$\frac{CE}{CB} = \frac{DE}{AB}, \quad (4)$$

or $$\frac{y}{700} = \frac{x}{800}, \quad (5)$$

and $$\frac{z}{600} = \frac{x}{800}. \quad (6)$$

Hence, $$y = \frac{7x}{8}, \quad (7)$$

$$z = \frac{3x}{4}. \quad (8)$$

From (7) and (8), $$x + y + z = \frac{21x}{8}. \quad (9)$$

From (9) and (2), $$2100 + 2x = 1800 + \frac{21x}{8}. \quad (10)$$

Hence, $$\frac{5x}{8} = 300. \quad (11)$$

$$x = 480. \quad (12)$$

From (12) and (7), $$y = 420. \quad (13)$$

From (12) and (8), $$z = 360. \quad (14)$$

The lengths of the sides are: $DE = 480$ yd., $EC = 360$ yd., and $CD = 420$ yd.

34. Solve Problem 33 if the division line runs parallel to the side of length 700 yd.

35. Solve Problem 33 if the division line runs parallel to the side of length 600 yd.

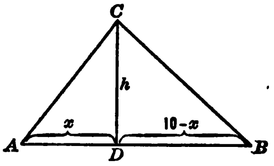
36. Solve Problem 33 if the sides are a, b, c, with the division line parallel to a, and the required length of fence, $2p$.

37. It is known that the sum of the three angles of any triangle is 180°. In a certain triangle the difference between the first angle and the second is 10°, and between the second angle and the third is 25°. Find the number of degrees in each angle.

38. The sum of the dimensions of a rectangular box is 13½ ft.; the height equals one half the sum of the length and breadth and also equals twice their difference. Find the dimensions.

QUADRATIC EQUATIONS

39. The sides of a triangle are $AC = 7$, $BC = 9$, and $AB = 10$. Calculate the length of the altitude on the side 10, and of the two segments into which the altitude divides that side.



Solution. Using the notations of the figure,

$$h^2 = 7^2 - x^2, \tag{1}$$

and
$$h^2 = 9^2 - (10 - x)^2. \tag{2}$$

Then,
$$7^2 - x^2 = 9^2 - (10 - x)^2, \tag{3}$$

or
$$7^2 = 9^2 - 100 + 20x. \tag{4}$$

Then,
$$68 = 20x, \tag{5}$$

and
$$x = \tfrac{17}{5}; \tag{6}$$

also,
$$10 - x = \tfrac{33}{5}. \tag{7}$$

By (1),
$$h^2 = 7^2 - (\tfrac{17}{5})^2. \tag{8}$$

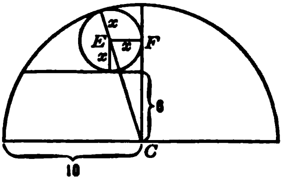
Then,
$$h = \frac{\sqrt{7^2 \cdot 5^2 - 17^2}}{5} = \frac{6\sqrt{26}}{5}. \tag{9}$$

The segments are $\frac{17}{5}$ and $\frac{33}{5}$ and the altitude is $\frac{6\sqrt{26}}{5}$.

40. Calculate similarly the length of the altitude on the side of length 7, and of the segments into which the altitude divides that side.

41. Calculate similarly the length of the altitude on the side of length 9, and of the segments into which the altitude divides that side.

42. If the lengths of the sides of a triangle are *a*, *b*, *c*, calculate the lengths of the segments into which each side is divided by the altitude on that side.



43. In a circle of radius 10, a chord is drawn at distance 6 from the center. Find the radius of a circle that is tangent to the circle, to the chord, and to a diameter perpendicular to it.

Solution. Using the notations of the figure:

$$\overline{EC}^2 = \overline{EF}^2 + \overline{FC}^2. \qquad (1)$$

$$(10 - x)^2 = x^2 + (6 + x)^2. \qquad (2)$$

$$100 - 20x + x^2 = x^2 + 36 + 12x + x^2. \qquad (3)$$

$$x^2 + 32x - 64 = 0. \qquad (4)$$

$$x = -\frac{32 \pm \sqrt{32^2 + 4 \cdot 64}}{2} \qquad (5)$$

$$= -\frac{32 \pm 16\sqrt{4 + 1}}{2} \qquad (6)$$

$$= -16 \pm 8\sqrt{5}. \qquad (7)$$

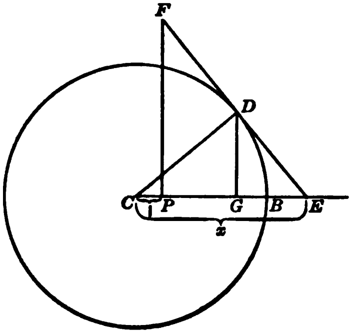
The negative value of x being inadmissible under the geometric conditions, we have:

$$x = -16 + 8\sqrt{5}. \qquad (8)$$

44. Solve the same problem, if the radius of the given circle is 12 ft. and the chord is 4 ft. from the center.

45. Solve Problem 43, if the radius of the given circle is r and the chord is at d distance from the center.

46. A point P is selected on a diameter of a circle of radius 6, at the distance 1 from the center. At P a perpendicular is erected to the diameter in question, and a tangent is drawn to the circle such that the point of contact of the tangent bisects the segment of the tangent lying between the perpendicular and the diameter produced. Find the distance from the center to the point where the tangent cuts the diameter produced.



Solution. Let
$$CE = x. \tag{1}$$

Then in the right triangle CDE,
$$\overline{DE}^2 = x^2 - 6^2. \tag{2}$$

From the similar triangle DCE and DGE,
$$\frac{DE}{CE} = \frac{GE}{DE}, \tag{3}$$

or
$$DE^2 = CE \cdot GE \tag{4}$$
$$= x \cdot GE. \tag{5}$$

Since D bisects FE,
$$GE = \frac{PE}{2} \tag{6}$$
$$= \frac{x-1}{2}. \tag{7}$$

From (2), (5), and (7),
$$\frac{x(x-1)}{2} = x^2 - 6^2, \tag{8}$$

or
$$x^2 - x = 2x^2 - 72. \tag{9}$$
$$\therefore x^2 + x - 72 = 0. \tag{10}$$
$$x = -\frac{1 \pm \sqrt{1+288}}{2} \tag{11}$$
$$= -9, 8. \tag{12}$$

The negative root also indicates a solution. It means that a second tangent satisfying the required conditions cuts the diameter produced on the opposite side from P, at the distance 9.

47. Solve the same problem if the radius is 12 and the point lies at the distance 2 from the center.

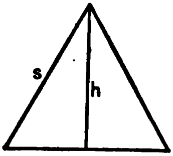
48. Solve the same problem if the radius is 15, and the distance of P from the center is 35.

49. Solve the same problem if the radius is r and the distance from the center is d.

50. The tangent to a circle is a mean proportional between the segments of the secant from the same point. Find the length of the tangent, if the segments of the secant are 4 ft. and 9 ft.

51. Two chords AB and CD intersect at O within the circle. The product of OA and OB equals the product of OC and OD. Given $OA = 4$, $OB = 8$, and $CD = 12$, find OC and OD.

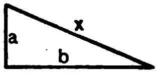
52. 4 times the square of the altitude (h) of an equilateral triangle equals 3 times the square of a side s. Express this by an equation. Solve the equation for s. Also for h in terms of s.



53. Find the altitude of an equilateral triangle whose side is 20 in., using 1.7321 as $\sqrt{3}$.

54. Find the side of an equilateral triangle, if $h = 9\sqrt{3}$ in.

55. The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. Express this relation in the form of an equation, using the letters in the figure.



56. Find the length of the hypotenuse of a right triangle the other two sides of which are 3 ft. and 4 ft. Also of one whose other two sides are 15 ft. and 20 ft.

57. Solve the equation $c^2 = a^2 + b^2$ for a; for b.

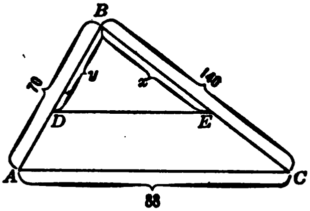
58. Find b in Exercise 57, if $c = 25$ and $a = 15$.

Similarly, determine the numbers to fill the blanks. To simplify calculation use the relation $c^2 - a^2 = (c - a)(c + a)$:

	59.	**60.**	**61.**	**62.**	**63.**	**64.**	**65.**
a.	15	7	40	45	—	208	44
b.	—	24	—	—	171	—	117
c.	17	—	41	53	221	233	—

SIMULTANEOUS QUADRATICS

66. The owner of a triangular lot whose sides are 70, 88, and 140 rd. in length wishes to divide it by a straight fence into two parts that shall be equal in area and also have the same perimeter. If the fence connects the sides of length 70 and 140 rd., how must it be placed?



Solution. Let DE be the desired position of the fence.

Then,
$$\triangle ABC = 2\triangle DBE. \qquad (1)$$

Since the triangles have one angle in common,
$$\frac{\triangle ABC}{\triangle DBE} = \frac{70 \cdot 140}{xy}, \qquad (2)$$

or, by (1),
$$2 = \frac{70 \cdot 140}{xy}, \qquad (3)$$

or
$$xy = 35 \cdot 140. \qquad (4)$$

By the conditions of the problem,
$$BD + BE + DE = DA + AC + CE + ED. \qquad (5)$$

Subtracting DE from both members and replacing the other lines by their values,
$$x + y = 70 - y + 140 - x + 88, \qquad (6)$$
$$2(x + y) = 298, \qquad (7)$$
$$x + y = 149. \qquad (8)$$

From (4),
$$4xy = 4 \cdot 35 \cdot 140 = \overline{140}^2. \qquad (9)$$

Squaring (8) and subtracting (9) from the result,
$$(x - y)^2 = \overline{149}^2 - \overline{140}^2 \qquad (10)$$
$$= (149 + 140)(149 - 140) \qquad (11)$$
$$= 289 \cdot 9. \qquad (12)$$

Then,
$$x - y = \pm 17 \cdot 3 = \pm 51. \qquad (13)$$

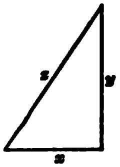
From (8) and (13),
$$x = 100,\ 49. \qquad (14)$$
$$y = 49,\ 100. \qquad (15)$$

These values satisfy the algebraic equations, but in the concrete problem $y = 100$ is inadmissible, since y lies on the side of length 70. Hence, in the concrete problem, the result is $x = 100$, $y = 49$.

67. Solve the same problem if the fence connects the sides of length 70 and 88.

68. Solve the same problem if the fence connects the sides of length 88 and 140.

69. Solve the same problem if the sides are of length a, b, c, and the fence connects the sides of lengths a and c.



70. Find the sides of a right-angled triangle, given its area 25, and its perimeter 30.

Solution. Let x, y, z denote the sides of the triangle, z being the hypotenuse.

Then,
$$x^2 + y^2 = z^2, \tag{1}$$
$$x + y + z = 30, \tag{2}$$
and
$$\frac{xy}{2} = 25. \tag{3}$$

Multiplying both members of (3) by 4,
$$2xy = 100. \tag{4}$$

Adding (4) and (1),
$$x^2 + 2xy + y^2 = z^2 + 100, \tag{5}$$
or
$$(x + y)^2 = z^2 + 100. \tag{6}$$
From (2),
$$x + y = 30 - z, \tag{7}$$
or
$$(x + y)^2 = (30 - z)^2. \tag{8}$$
From (8) and (6),
$$(30 - z)^2 = z^2 + 100. \tag{9}$$
$$900 - 60z + z^2 = z^2 + 100. \tag{10}$$
$$60z = 800. \tag{11}$$
$$z = \tfrac{40}{3}. \tag{12}$$
From (7),
$$x + y = \tfrac{50}{3}, \tag{13}$$
or
$$x^2 + 2xy + y^2 = \tfrac{2500}{9}. \tag{14}$$

Multiplying (4) by 2 and subtracting the result from (14),
$$x^2 - 2xy + y^2 = \tfrac{100}{9}, \tag{15}$$
or
$$x - y = \pm\frac{10\sqrt{7}}{3}. \tag{16}$$

Adding (13) and (16) and dividing the result by 2,
$$x = \frac{25 \pm 5\sqrt{7}}{3}. \tag{17}$$

Subtracting (16) from (13) and dividing the result by 2,
$$y = \frac{25 \mp 5\sqrt{7}}{3}. \tag{18}$$

The sides are $\dfrac{25 + 5\sqrt{7}}{3}$, $\dfrac{25 - 5\sqrt{7}}{3}$, and $\dfrac{40}{3}$.

71. Solve the same problem if the area of the triangle is 64, and the perimeter 48.

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